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ON THE RIGIDITY OF WEBS

BY MICHEL BELLIART

ABSTRACT. — Plane d-webs have been studied a lot since their appearance at the turn of the 20^{th} century. A rather recent and striking result for them is the theorem of Dufour, stating that the measurable conjugacies between 3-webs have to be analytic. Here, we show that even the set-theoretic conjugacies between two d-webs, $d \geq 3$ are analytic unless both webs are analytically parallelizable. Between two set-theoretically conjugate parallelizable d-webs, however, there always exists a nonmeasurable conjugacy; still, every pair of set-theoretically conjugate 3-webs (parallelizable or not) also are analytically conjugate, while if $d \geq 4$ there exist pairs of d-webs which are set-theoretically conjugate but not even measurably so.

RÉSUMÉ (*Sur la rigidité des tissus*). — Les d-tissus plans ont été amplement étudiés depuis leur apparition au début du XX^{e} siècle. Un résultat relativement récent et impressionnant est le théorème de Dufour qui stipule que les conjugaisons mesurables entre 3-tissus sont nécessairement analytiques. Dans cet article nous montrons que les conjugaisons ensemblistes entre d-tissus (avec $d \geq 3$) sont analytiques sauf si les deux tissus sont analytiquement parallélisables. Cependant, entre deux d-tissus parallélisables conjugués de manière ensembliste il existe toujours une conjugaison nonmesurable ; de plus, toute paire de 3-tissus conjugués de manière ensembliste (qu'ils soient parallélisables ou non) sont également conjugués analytiquement, alors que si $d \geq 4$, il existe des paires de d-tissus qui sont conjugués de manière ensembliste mais non pas de manière mesurable.

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1. Introduction

1.1. Foliations. — Throughout this note we consider the word "foliation" as meaning "analytic foliation of the plane $\mathbb{R}^2$ by curves". The classification of foliations (in this very restricted sense of the word) has been known for years; first of all, we can describe foliations in two dual ways:

- Given any foliation $\mathcal{F}$, there exists an analytic action of $\mathbb{R}$ on $\mathbb{R}^2$ whose orbits are the leaves of $\mathcal{F}$ (this is not difficult at all to show).
- Given any foliation $\mathcal{F}$, there exists a continuous submersion ϕ from $\mathbb{R}^2$ to $\mathbb{R}$ which is constant along the leaves of $\mathcal{F}$: therefore these leaves are the connected components of the fibers of ϕ (this is Kaplan's Theorem [9]).

Starting from there, it is not too difficult to describe the topological conjugacy classes of foliations; see [7] for a nice description of the classification (due to Kaplan [10]).

We simply define the set-theoretic conjugacies between two foliations $\mathcal{F}$ and $\mathcal{F}'$ as being bijective maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which send every leaf of $\mathcal{F}$ bijectively onto some leaf of $\mathcal{F}'$. This is a straightforward but completely formal definition; observe that any two foliations are set-theoretically conjugate! But the following very simple example shows us that the notion of a set-theoretic conjugacy already stops being trivial if we consider more than just one foliation at a time.

Example 1.1.1. — Let $\mathcal{F}_1$ be the foliation of $\mathbb{R}^2$ by horizontal lines; let $\mathcal{F}_2$ be that by vertical lines, and let $\mathcal{F}_3$ be that by curves having the form $y = \mathrm{e}^x + C$ where (x, y) are the natural coordinates and C is a parameter. Then, there is no bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ inducing a conjugacy of $\mathcal{F}_1$ onto itself and a conjugacy of $\mathcal{F}_2$ onto $\mathcal{F}_3$ at the same time: indeed any leaf of $\mathcal{F}_1$ and any leaf of $\mathcal{F}_2$ meet at exactly one point, while for every leaf L of $\mathcal{F}_3$ there is on the contrary a leaf of $\mathcal{F}_1$ not meeting L.

1.2. Webs. — Funnily enough, webs have appeared in mathematics way before foliations, perhaps because their local geometry is so visibly richer. We call *d-web* the datum $\mathcal{W} = (\mathcal{F}_1, \ldots, \mathcal{F}_d)$ of $d \geq 3$ foliations which we require to be pairwise transverse at each point: this means *e.g.*, that any leaf L_1 of $\mathcal{F}_1$ and any leaf L_2 of $\mathcal{F}_2$, if not disjoint, intersect at exactly one point with distinct tangents.

We should mention that usually, a web is *locally* defined as an *unordered* collection of d foliations (possibly singular) which are in general position. There are then obstructions to the possibility of "separating" this data in d distinct foliations; but these obstructions read either on the singular locus of the configuration, or on the topology of the ambient manifold. Here, we chose to work with everywhere transverse nonsingular (local) foliations of a contractible space; for this reason, our d local foliations glue into global ones and we can order the collection of them.

In 1908 already, Cartan [3] asks *to study the topology of the figure formed by three families of curves*, a problem which he calls the *first problem of textile geometry*. From the 1930ies on, the school of Blashke and Bol will give this problem due consideration, as well as generalize it a lot; the works of that school are collected in the books [2] and [1]. For a more modern point of view on web theory, we should mention the surveys [4] and [13] written by two experts in the field.

We say, of course, that the bijection f of $\mathbb{R}^2$ conjugates the d-web $\mathcal{W}$ to the d-web $\mathcal{W}'$ if it conjugates each of the foliations forming $\mathcal{W}$ to the foliation of $\mathcal{W}'$ which has the same index. Example 1.1.1 implies that not every couple of d-webs are conjugate in this way. Finally, we call a web *parallelizable* if it is conjugate to some web whose foliations are by parallel affine lines.

1.3. Rigidity. — We have grown used to the fact that a typical map from $\mathbb{R}$ to $\mathbb{R}$ is not measurable, that a typical measurable such map is not continuous, and so forth... But the first historical examples of such uncanny behaviour had been built as counterexamples and did not answer any "real" problems: for this reason, even after these first examples appeared in the works of Peano, Riemann, Weierstraß *et al.*, one could still place some faith in the following informal, sadly erroneous belief,

Credo 1.3.1. — If a problem whose datum is purely analytic has a unique solution, then that solution must be analytic.

As we hinted to, the above credo was baffled by many counterexamples which, as a rule, came from dynamical systems theory: for instance, to certain dynamical systems – the so-called *Anosov diffeomorphi sms* – one can associate invariant foliations which will not necessarily be differentiable even if the diffeomorphism we started with is analytic (see [11, III.3]; we quote from the same source: "in general, even if f is C^∞, Poincaré transformations are not even Lipschitz"). Another famous example is that of the "cohomological equation" $f(x+k) - f(x) = g(x)$, where the datum g and the unknown f both are smooth functions from $\mathbb{R}$ to itself with period one and k is a given real number: as soon as k is irrational and $\int_0^1 g(x)\mathrm{d}x = 0$, there exists a formal solution for that equation in the shape of a Fourier series $\widehat{f}$ which, in general, does not converge at all; for $\widehat{f}$ to converge to a *smooth* function whatever our choice of g, the real number k must be diophantine (see [8]). In conclusion, examples of analytic problems whose solutions only possess a low regularity abound in modern practice, and this is what makes the following result so very interesting to us:

THEOREM (Dufour, [6]). — *Any measurable bijection of $\mathbb{R}^2$ onto itself which exchanges two 3-webs must be analytic.*

This is the first version of Dufour's Theorem; we should perhaps mention the existence of a recent generalization of it to arbitrary dimensions and codimensions by the same author, and the existence of a complex version by Nakai in codimension one (see [5] and [12], respectively).

1.4. Statement of the theorems. — The purpose of this note is mostly to show:

THEOREM 1.4.1. — *Let $\mathcal{W}$ and $\mathcal{W}'$ be two d-webs one of which at least is not analytically parallelizable. Then every set-theoretic conjugacy from $\mathcal{W}$ to $\mathcal{W}'$ is analytic.*

If $\mathcal{W}$ is a parallelizable web, one can easily show that $\mathcal{W}$ possesses non-measurable self-automorphisms; so, Theorem 1.4.1 is sharp. We should next wonder if the existence of a set-theoretic conjugacy f between two d-webs always implies the existence of an analytic one f' *not necessarily equal to* f: this is a weaker sort of rigidity. Theorem 1.4.1 already solved that problem except for parallelizable webs, for which we have

THEOREM 1.4.2. — *The two notions of analytic conjugacy and of set-theoretic conjugacy for parallelizable d-webs are equivalent precisely if $d = 3$.*

For $d \geq 4$ we will exhibit interesting counterexamples.

We would like to underline the similarity of ideas between our proof of Theorem 1.4.1 and the so-called *fundamental theorem of affine geometry*. This famous result states that a bijection of $\mathbb{R}^2$ preserving the family of lines has to be affine; recall how the proof works: first, by using elementary constructions which are in fact valid over any field $\mathbb{K}$ but the field $\{0,1\}$, one associates to any line-preserving bijection f of $\mathbb{K}^2$ a field automorphism τ_f of $\mathbb{K}$ such that the equality $f(\sum m_\imath M_\imath) = \sum \tau_f(m_\imath)M_\imath$ holds for any barycenter $\sum m_\imath M_\imath$ in $\mathbb{K}^2$. Thus, the map f will be affine precisely if the field automorphism τ_f is trivial... which, as one knows, must happen if $\mathbb{K} = \mathbb{R}$. Now, the construction of τ_f uses the whole set of lines in $\mathbb{K}^2$; but if we restrict our attention to three particular families of parallel lines, we can still build an automorphism of abelian group τ_f of $\mathbb{K}^2$ and show that the equality $f(M + \vec{v}) = f(M) + \tau_f(\vec{v})$ holds for every point M and vector $\vec{v}$. This equality is a weakening of the former one obtained thanks to the whole set of lines, if we remember that formally, a vector is simply a barycenter with total mass zero. Instead of the fact that every field automorphism of $\mathbb{R}$ is trivial, we may now invoke the other fact that every *measurable* group automorphism of $\mathbb{R}^2$ is linear to obtain