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SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

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ABSTRACT. — Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Given a semistable vector bundle E over X , we show that its direct image F_*E under the Frobenius map F of X is again semistable. We deduce a numerical characterization of the stable rank- p vector bundles F_*L , where L is a line bundle over X .

RÉSUMÉ (*Semi-stabilité des images directes par Frobenius sur les courbes*)

Soit X une courbe projective lisse de genre ≥ 2 définie sur un corps k algébriquement clos de caractéristique $p > 0$. Étant donné un fibré vectoriel semi-stable E sur X , nous montrons que l'image directe F_*E par le morphisme de Frobenius F de X est aussi semi-stable. Nous déduisons une caractérisation numérique du fibré vectoriel stable F_*L de rang p , où L est un fibré en droites sur X .

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$ and let $F : X \rightarrow X_1$ be the relative k -linear Frobenius map. It is by now a well-established fact that on

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any curve X there exist semistable vector bundles E such that their pull-back under the Frobenius map F^*E is not semistable [4, 5]. In order to control the degree of instability of the bundle F^*E , one is naturally lead (through adjunction $\mathrm{Hom}_{\mathcal{O}_X}(F^*E, E') = \mathrm{Hom}_{\mathcal{O}_{X_1}}(E, F_*E')$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

THEOREM 1.1. — *Assume that $g \geq 2$. If E is a semistable vector bundle over X (of any degree), then F_*E is also semistable.*

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that F_*L is stable for a line bundle L (see [4, Proposition 1.2]) and that in small characteristics the bundle F_*E is stable for any stable bundle E of small rank (see [3]). The main ingredient of the proof is Faltings' cohomological criterion of semistability. We also need the fact that the generalized Verschiebung V , defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank- r vector bundles over X_1 with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map F ,

$$V_r : \mathcal{M}_{X_1}(r) \dashrightarrow \mathcal{M}_X(r), \quad E \longmapsto F^*E$$

is dominant for large r . We actually show a stronger statement for large r .

PROPOSITION 1.2. — *If $\ell \geq g(p-1)+1$ and ℓ prime, then the generalized Verschiebung V_ℓ is generically étale for any curve X . In particular V_ℓ is separable and dominant.*

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant ν of a vector bundle E , defined as

$$\nu(E) := \mu_{\max}(F^*E) - \mu_{\min}(F^*E),$$

where $\mu_{\max}$ (resp. $\mu_{\min}$) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of F^*E .

PROPOSITION 1.3. — *For any semistable rank- r vector bundle E*

$$\nu(E) \leq \min((r-1)(2g-2), (p-1)(2g-2)).$$

We note that the inequality $\nu(E) \leq (r-1)(2g-2)$ was proved in [10, Corollary 2], and in [11, Theorem 3.1]. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (= number of pieces) of the Harder-Narasimhan filtration of F^*E is at most p for semistable E .

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant ν .

PROPOSITION 1.4. — *Let E be a stable rank- p vector bundle over X . Then the following statements are equivalent.*

- 1) *There exists a line bundle L such that $E = F_*L$.*
- 2) $\nu(E) = (p-1)(2g-2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case $\mu(E) = g - 1$

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles E with slope $\mu(E) = g - 1$.

Let E be a semistable vector bundle over X of rank r and let s be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}.$$

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \rightarrow X_1$, we obtain

$$\mu(F_*E) = g - 1 + \frac{s}{pr}.$$

Let $\pi : \widetilde{X} \rightarrow X$ be a connected étale covering of degree n and let $\pi_1 : \widetilde{X}_1 \rightarrow X_1$ denote its twist by the Frobenius of k (see [9, Section 4]). The diagram

$$(2.1) \quad \begin{array}{ccc} \widetilde{X} & \xrightarrow{F} & \widetilde{X}_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ X & \xrightarrow{F} & X_1 \end{array}$$

is Cartesian and we have an isomorphism

$$\pi_1^*(F_*E) \cong F_*(\pi^*E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that π^*E is semistable. Moreover if $F_*(\pi^*E)$ is semistable, then F_*E is also semistable.

Let L be a degree d line bundle over $\widetilde{X}_1$. The projection formula

$$F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L$$

shows that semistability of $F_*(\pi^*E)$ is equivalent to semistability of

$$F_*(\pi^*E \otimes F^*L).$$

Let $\tilde{g}$ denote the genus of $\tilde{X}$. By the Riemann-Hurwitz formula, one has $\tilde{g} - 1 = n(g - 1)$. We compute

$$\mu(\pi^*E \otimes F^*L) = n(g - 1) + n\frac{s}{r} + pd = \tilde{g} - 1 + n\frac{s}{r} + pd,$$

which gives

$$\mu(F_*(\pi^*E \otimes F^*L)) = \tilde{g} - 1 + n\frac{s}{pr} + d.$$

LEMMA 2.1. — *For any integer m there exists a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$ for some k .*

Proof. — If the p -rank of X is nonzero, the statement is clear. If the p -rank is zero, we know by [9, Corollaire 4.3.4], that there exist connected étale coverings $Y \rightarrow X$ of degree p^t for infinitely many integers t (more precisely for all t of the form $(\ell - 1)(g - 1)$ where ℓ is a large prime). Now we decompose $m = p^s u$ with p not dividing u . We then take a covering $Y \rightarrow X$ of degree p^t with $t \geq s$ and a covering $\tilde{X} \rightarrow Y$ of degree u . $\square$

Now the lemma applied to the integer $m = pr$ shows existence of a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$. Hence $n\frac{s}{pr}$ is an integer and we can take d such that $n\frac{s}{pr} + d = 0$.

To summarize, we have shown that for any semistable E over X there exists a covering $\pi : \tilde{X} \rightarrow X$ and a line bundle L over $\tilde{X}_1$ such that the vector bundle $\tilde{E} := \pi^*E \otimes F^*L$ is semistable with $\mu(\tilde{E}) = \tilde{g} - 1$ and such that semistability of $F_*\tilde{E}$ implies semistability of F_*E .

3. Proof of Theorem 1.1

In order to prove semistability of F_*E we shall use the cohomological criterion of semistability due to Faltings [2].

PROPOSITION 3.1 (see [6, Théorème 2.4]). — *Let E be a rank- r vector bundle over X with $\mu(E) = g - 1$ and ℓ an integer $> \frac{1}{4}r^2(g - 1)$. Then E is semistable if and only if there exists a rank- ℓ vector bundle G with trivial determinant such that*

$$h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.$$

Moreover if the previous condition holds for one bundle G , it holds for a general bundle by upper semicontinuity of the function $G \mapsto h^0(X, E \otimes G)$.

REMARK. — The proof of this proposition (see [6, Section 2.4]) works over any algebraically closed field k .