

Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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**Tome 135
Fascicule 1**

2007

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du Centre national de la recherche scientifique

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ON A CERTAIN GENERALIZATION OF SPHERICAL TWISTS

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ABSTRACT. — This note gives a generalization of spherical twists, and describe the autoequivalences associated to certain non-spherical objects. Typically these are obtained by deforming the structure sheaves of $(0, -2)$ -curves on threefolds, or deforming $\mathbb{P}$ -objects introduced by D. Huybrechts and R. Thomas.

RÉSUMÉ (*Sur une généralisation des twists sphériques*). — Cette note donne une généralisation des twists sphériques et décrit des auto-équivalences associées à certains objets qui ne sont pas sphériques. Typiquement ces objets sont obtenus par déformation du faisceau structural d'une $(0, 2)$ -courbe dans une variété de dimension trois ou d'un $\mathbb{P}$ -objet introduit par D. Huybrechts et R. Thomas.

1. Introduction

We introduce a new class of autoequivalences of derived categories of coherent sheaves on smooth projective varieties, which generalizes the notion of spherical twists given in [12]. Such autoequivalences are associated to a certain class of objects, which are not necessary spherical but are interpreted as “fat”

Texte reçu le 4 avril 2006, révisé le 23 mai 2006

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2000 Mathematics Subject Classification. — 18E30, 14J32.

Key words and phrases. — Derived categories, mirror symmetries.

version of them. We introduce the notion of R -spherical objects for a noetherian and artinian local $\mathbb{C}$ -algebra R , and imitate the construction of spherical twists to give the associated autoequivalences.

Let X be a smooth complex projective variety, and $D(X)$ be a bounded derived category of coherent sheaves on X . When X is a Calabi-Yau 3-fold, $D(X)$ is considered to represent the category of D -branes of type B , and should be equivalent to the derived Fukaya category on a mirror manifold under Homological mirror symmetry [8]. On the mirror side, there are typical symplectic automorphisms by taking Dehn twists along Lagrangian spheres [11]. The notions of spherical objects and associated twists were introduced in [12] in order to realize Dehn twists under mirror symmetry. Recall that $E \in D(X)$ is called *spherical* if the following holds [12]:

- $\mathrm{Ext}_X^i(E, E) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } i = \dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

Then one can construct the autoequivalence $T_E: D(X) \rightarrow D(X)$ which fits into the distinguished triangle [12]:

$$\mathbb{R}\mathrm{Hom}(E, F) \otimes_{\mathbb{C}} E \longrightarrow F \longrightarrow T_E(F),$$

for $F \in D(X)$. The autoequivalence T_E is called a *spherical twist*. This is a particularly important class of autoequivalences, especially when we consider A_n -configurations on surfaces as indicated in [7]. On the other hand, it has been observed that there are some autoequivalences which are not described in terms of spherical twists. This occurs even in the similar situation discussed in [7] as follows. Let $X \rightarrow Y$ be a three dimensional flopping contraction which contracts a rational curve $C \subset X$, and $X^\dagger \rightarrow Y$ be its flop. Then one can construct the autoequivalence [1, 3, 4],

$$\Phi := \Phi_{X^\dagger \rightarrow X}^{\mathcal{O}_{X \times_Y X^\dagger}} \circ \Phi_{X \rightarrow X^\dagger}^{\mathcal{O}_{X \times_Y X^\dagger}} : D(X) \longrightarrow D(X^\dagger) \longrightarrow D(X).$$

If $C \subset X$ is not a $(-1, -1)$ -curve, Φ is not written as a spherical twist, and our motivation comes from describing such autoequivalences. Let R be a noetherian and artinian local $\mathbb{C}$ -algebra. We introduce the notion of R -spherical objects defined on $D(X \times \mathrm{Spec} R)$. In the above example, $\mathrm{Spec} R$ is taken to be the moduli space of $\mathcal{O}_C(-1)$, and the universal family gives the R -spherical object. Our main theorem is the following:

THEOREM 1.1. — *To any R -spherical object $\mathcal{E} \in D(X \times \mathrm{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}}: D(X) \rightarrow D(X)$, which fits into the distinguished triangle*

$$\mathbb{R}\mathrm{Hom}_X(\pi_* \mathcal{E}, F) \stackrel{\mathbb{L}}{\otimes}_R \pi_* \mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. Here $\pi: X \times \operatorname{Spec} R \rightarrow X$ is the projection.

Using the notion of R -spherical objects and associated twists, we can also give the deformations of $\mathbb{P}$ -twists in the case which is not treated in [5].

Acknowledgement. — The author thanks Tom Bridgeland for useful discussions and comments. He is supported by Japan Society for the Promotion of Sciences Research Fellowships for Young Scientists, No 1611452.

Notations and conventions

- For a variety X , we denote by $D(X)$ its bounded derived category of coherent sheaves.
- Δ means the diagonal $\Delta \subset X \times X$ or the diagonal embedding $\Delta: X \rightarrow X \times X$.
- For another variety Y and an object $\mathcal{P} \in D(X \times Y)$, denote by $\Phi_{X \rightarrow Y}^{\mathcal{P}}$ the integral transform with kernel $\mathcal{P}$, i.e.,

$$\Phi_{X \rightarrow Y}^{\mathcal{P}}(*) := \mathbb{R}p_{Y*}(p_X^*(*) \overset{\mathbb{L}}{\otimes} \mathcal{P}): D(X) \longrightarrow D(Y).$$

Here p_X, p_Y are projections from $X \times Y$ onto corresponding factors.

2. Generalized spherical twists

Let X be a smooth projective variety over $\mathbb{C}$ and R be a noetherian and artinian local $\mathbb{C}$ -algebra. We introduce the notion of R -spherical objects defined on $D(X \times \operatorname{Spec} R)$. Let $\pi: X \times \operatorname{Spec} R \rightarrow X$ and $\pi': X \times \operatorname{Spec} R \rightarrow \operatorname{Spec} R$ be projections and $0 \in \operatorname{Spec} R$ be the closed point.

DEFINITION 2.1. — An object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$ is called *R -spherical* if the following conditions hold:

- $\mathcal{E}$ is represented by a bounded complex $\mathcal{E}^\bullet$ with each $\mathcal{E}^i$ a coherent $\mathcal{O}_{X \times \operatorname{Spec} R}$ -module flat over R . In particular we have the bounded derived restriction $E := \mathcal{E}^\bullet|_{X \times \{0\}} \in D(X)$.
- $\operatorname{Ext}_X^i(E, E) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } i = \dim X, \\ 0 & \text{otherwise;} \end{cases}$
- $E \otimes \omega_X \cong E$.

REMARK 2.2. — If $R = \mathbb{C}$, then R -spherical objects coincide with usual spherical objects.

We imitate the construction of the spherical twists in the following theorem.

THEOREM 2.3. — *To any R -spherical object $\mathcal{E} \in D(X \times \operatorname{Spec} R)$, we can associate the autoequivalence $T_{\mathcal{E}}: D(X) \rightarrow D(X)$, which fits into the distinguished triangle:*

$$\mathbb{R} \operatorname{Hom}_X(\pi_* \mathcal{E}, F) \stackrel{\mathbb{L}}{\otimes}_R \pi_* \mathcal{E} \longrightarrow F \longrightarrow T_{\mathcal{E}}(F),$$

for $F \in D(X)$. Here R -module structures on $\mathbb{R} \operatorname{Hom}_X(\pi_* \mathcal{E}, F)$ and $\pi_* \mathcal{E}$ are inherited from R -module structure on $\mathcal{E}$.

Proof. — First we construct the kernel of $T_{\mathcal{E}}$. Let p_{ij} and p_i be projections as in the following diagram

$$\begin{array}{ccccc} & & X \times \operatorname{Spec} R \times X & & \\ & \swarrow p_{12} & \downarrow p_{13} & \searrow p_{23} & \\ X \times \operatorname{Spec} R & & X \times X & & X \times \operatorname{Spec} R, \\ \downarrow \pi & \swarrow p_1 & & \searrow p_2 & \downarrow \pi \\ X & & & & X \end{array}$$

and consider the object

$$\mathcal{Q} := \mathbb{R} p_{13*} (p_{12}^* (\pi^! \mathcal{O}_X \stackrel{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \stackrel{\mathbb{L}}{\otimes} p_{23}^* \mathcal{E}) \in D(X \times X).$$

Here $\check{\mathcal{E}}$ means its derived dual. Then for $F \in D(X)$, we can calculate $\Phi_{X \rightarrow X}^{\mathcal{Q}}(F)$ as follows:

$$\begin{aligned} \Phi_{X \rightarrow X}^{\mathcal{Q}}(F) &\cong \mathbb{R} p_{2*} (\mathbb{R} p_{13*} (p_{12}^* (\pi^! \mathcal{O}_X \stackrel{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \stackrel{\mathbb{L}}{\otimes} p_{23}^* \mathcal{E}) \stackrel{\mathbb{L}}{\otimes} p_1^* F) \\ &\cong \mathbb{R} p_{2*} \mathbb{R} p_{13*} (p_{12}^* (\pi^! \mathcal{O}_X \stackrel{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \stackrel{\mathbb{L}}{\otimes} p_{23}^* \mathcal{E} \stackrel{\mathbb{L}}{\otimes} p_{13}^* p_1^* F) \\ &\cong \pi_* \mathbb{R} p_{23*} (p_{12}^* (\pi^! \mathcal{O}_X \stackrel{\mathbb{L}}{\otimes} \check{\mathcal{E}}) \stackrel{\mathbb{L}}{\otimes} p_{23}^* \mathcal{E} \stackrel{\mathbb{L}}{\otimes} p_{12}^* \pi^* F) \\ &\cong \pi_* \{ \mathcal{E} \stackrel{\mathbb{L}}{\otimes} \mathbb{R} p_{23*} p_{12}^* (\pi^! \mathcal{O}_X \stackrel{\mathbb{L}}{\otimes} \check{\mathcal{E}} \stackrel{\mathbb{L}}{\otimes} \pi^* F) \} \\ &\cong \pi_* \{ \mathcal{E} \stackrel{\mathbb{L}}{\otimes} \pi'^* \mathbb{R} \pi'_* \mathbb{R} \operatorname{Hom}(\mathcal{E}, \pi^! F) \} \\ &\cong \pi_* \mathcal{E} \stackrel{\mathbb{L}}{\otimes}_R \mathbb{R} \operatorname{Hom}(\pi_* \mathcal{E}, F). \end{aligned}$$

The fifth equality comes from the base change formula for the diagram below:

$$\begin{array}{ccc} X \times \operatorname{Spec} R \times X & \xrightarrow{p_{12}} & X \times \operatorname{Spec} R \\ \downarrow p_{23} & & \downarrow \pi' \\ X \times \operatorname{Spec} R & \xrightarrow{\pi'} & \operatorname{Spec} R. \end{array}$$