THE NUMBER OF CONJUGACY CLASSES OF THE CREMONA GROUP

Jérémy Blanc

THE NUMBER OF CONJUGACY CLASSES OF ELEMENTS OF THE CREMONA GROUP OF SOME GIVEN FINITE ORDER

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Abstract. — This note presents the study of the conjugacy classes of elements of some given finite order n in the Cremona group of the plane. In particular, it is shown that the number of conjugacy classes is infinite if n is even, $n = 3$ or $n = 5$, and that it is equal to 3 (respectively 9) if $n = 9$ (respectively if $n = 15$) and to 1 for all remaining odd orders. Some precise representative elements of the classes are given.

Résumé (*Le nombre de classes de conjugaison du groupe de Cremona d'un ordre fini donné*)

Cet article présente l'étude des classes de conjugaisons des éléments d'ordre fini n dans le groupe de Cremona du plan. En particulier, il est montré que le nombre de classes de conjugaisons est infini si n est pair, $n = 3$ ou $n = 5$, et que ce nombre est égal à 3 (respectivement 9) si $n = 9$ (respectivement si $n = 15$) et à 1 pour les nombres entiers impairs restant. Des représentants explicites des classes de conjugaisons sont donnés.

1. Introduction

Let us recall that a *rational transformation* of $\mathbb{P}^2(\mathbb{C})$ is a map of the form

$$(x : y : z) \dashrightarrow (\varphi_1(x, y, z) : \varphi_2(x, y, z) : \varphi_3(x, y, z)),$$

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where $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{C}[x, y, z]$ are homogeneous polynomials of the same degree. If such a map has an inverse of the same type, we say that it is *birational*.

The *Cremona group* is the group of birational transformations of $\mathbb{P}^2(\mathbb{C})$. This group has been studied since the 19th century by many mathematicians. One of the first natural questions that we may ask when we study some group is the following:

QUESTION 1.1. — *Given some positive integer n, how many conjugacy classes of elements of order n exist in the Cremona group?*

First of all, it is important to note that the number of conjugacy classes is at least one, for any integer n, as the linear automorphism

$$(x : y : z) \longmapsto (x : y : \mathrm{e}^{2\mathbf{i}\pi/n}z)$$

is a representative element of one class. It was proved in [2] that all the linear automorphisms of the plane of the same finite order are birationally conjugate (the same is true in any dimension, see [4, Prop. 5]); to find more conjugacy classes we have therefore to show the existence of non-linearizable birational transformations.

The first answer to Question 1.1 was given in [3] for $n = 2$. Infinitely many involutions which are not conjugate are found. Since the proof of [3] is considered as incomplete, a precise and complete one may be found in [1].

In [9], the answer for n prime is given. It is shown that the number of conjugacy classes is infinite for $n = 3, 5$ and is equal to 1 if n is a prime integer ≥ 7.

For other orders, a lot of examples have been given in the ancient articles (for example in [12], [16]) and in many more recent articles, the most recent one being [8]. However, the precise answer to Question 1.1 was not given for n not prime.

In this paper, we answer Question 1.1 for any integer n, proving the following theorems:

THEOREM 1.2. — *For any even integer n, the number of conjugacy classes of elements of order n in the Cremona group is infinite. This is also true for $n = 3, 5$.*

THEOREM 1.3. — *For any odd integer $n \neq 3, 5$, the number of conjugacy classes of elements of order n in the Cremona group is finite. Furthermore this number is equal to 3 (respectively 9) if $n = 9$ (respectively if $n = 15$) and is 1 otherwise.*

This paper is a part of the author's PHD Thesis (the full text is available in [5], the results without proofs have been published in [6]). The results have been a little improved and the arguments have been slightly ameliorated to reduce the length of the proofs. We thank a lot our PHD advisor Thierry Vust, and also Arnaud Beauville, Igor Dolgachev, Ivan Pan and Felice Ronga for helpful discussions.

REMARK 1.4. — Theorem 1.2 and a part of Theorem 1.3 may be proved using the classification of finite Abelian subgroups of the Cremona group made in [5], but the proof is really long and intricate, whereas the results of this paper only require a much shorter and direct proof. Moreover, the precise counting of Theorem 1.3 does not follow from the classification (in general, it is not easy to decide whether or not two elements of the same subgroup are conjugate).

2. Automorphisms of rational surfaces

Let us remark the obvious but important observation: take some birational transformation φ of a rational surface S. Any birational map $\lambda : S \dashrightarrow \mathbb{P}^2$ conjugates φ to the birational transformation $\varphi_\lambda = \lambda \circ \varphi \circ \lambda^{-1}$ of $\mathbb{P}^2$. Although φ_λ is not unique, all the possible φ_λ's form an unique conjugacy class of birational transformations of $\mathbb{P}^2$.

Conversely, taking some birational transformation of $\mathbb{P}^2$, we may conjugate it to a birational transformation of any rational surface. If the order of the transformation is finite, we may furthermore conjugate it to a (*biregular*) *automorphism* of a rational surface. (See for example [10], Theorem 1.4.)

An important family of rational surfaces are the rational surfaces with an ample anticanonical divisor, i.e. the *Del Pezzo surfaces*. These surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$, and the blow-up of $1 \leq r \leq 8$ points of $\mathbb{P}^2$ in a general position (i.e. such that no irreducible curve of self-intersection -2 belongs to the surface). There is an extensive literature about this; some descriptions may be found for example in [13]. The *degree* of such a surface is the square of its canonical divisor, and is an integer between 1 and 9; it is 9 for $\mathbb{P}^2$, 8 for $\mathbb{P}^1 \times \mathbb{P}^1$ and $9 - r$ for the blow-up of r points in $\mathbb{P}^2$. Almost all of our examples of rational surfaces will be Del Pezzo surfaces.

3. Elements of order 3,5 and of any even order.
The proof of Theorem 1.2

Let us give families of conjugacy classes of elements of order 2, 3 and 5 of the Cremona group.

EXAMPLE 3.1 (Birational transformations of order 2). — Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ in $\mathbb{C}$ be all distinct. The birational map

$$((x_1 : x_2), (y_1 : y_2)) \dashrightarrow \left((x_1 : x_2), \left(y_2 \prod_{i=1}^{n}(x_1 - b_i x_2) : y_1 \prod_{i=1}^{n}(x_1 - a_i x_2) \right) \right)$$

of $\mathbb{P}^1 \times \mathbb{P}^1$ is an involution, which is classically called *de Jonquières involution*. Its fixed points form a smooth curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ of equation

$$(y_1)^2 \cdot \prod_{i=1}^{n}(x_1 - a_i x_2) = (y_2)^2 \cdot \prod_{i=1}^{n}(x_1 - b_i x_2).$$

The restriction to Γ of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ on the first factor is a surjective morphism $\Gamma \to \mathbb{P}^1$ of degree 2, ramified over the points

$$(a_1 : 1), \ldots, (a_n : 1), (b_1 : 1), \ldots, (b_n : 1).$$

The curve Γ is therefore an *hyperelliptic curve*.

These involutions are birationally equivalent to those of [1], *Example* 2.4 (c).

EXAMPLE 3.2 (Birational transformations of order 3). — Let F be a non-singular form of degree 3 in three variables and let

$$\Gamma = \left\{ (x : y : z) \in \mathbb{P}^2 \mid F(x, y, z) = 0 \right\}$$

be the smooth cubic plane curve associated to it. The surface

$$S = \left\{ (w : x : y : z) \in \mathbb{P}^3 \mid w^3 = F(x, y, z) \right\} \subset \mathbb{P}^3$$

is thus a smooth cubic surface in $\mathbb{P}^3$, which is rational (it is a Del Pezzo surface of degree 3, see for example [13], Theorem III.3.5). The map $w \mapsto e^{2i\pi/3} w$ gives rise to an automorphism of S whose set of fixed points is isomorphic to the elliptic curve Γ.

Such elements generate cyclic groups of order 3, *already given in* [9], *Theorem A, case A1.*

EXAMPLE 3.3 (Birational transformations of order 5). — Let us choose λ, μ in $\mathbb{C}$ such that the surface

$$S = \left\{ (w : x : y : z) \in \mathbb{P}(3, 1, 1, 2) \mid w^2 = z^3 + \lambda x^4 z + x(\mu x^5 + y^5) \right\}$$

is smooth. The surface S is thus rational (it is a Del Pezzo surface of degree 1, see [13, Thm. III.3.5]) and the map $y \mapsto e^{2i\pi/5} y$ gives rise to an automorphism of S whose set of fixed points is the union of the point $(0 : 0 : 1 : 0)$ and the elliptic curve which is the trace on S of the equation $y = 0$.

The corresponding cyclic groups of order 5 *were given in* [9], *Theorem A, case A3.*