Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

FUNDAMENTAL THEOREM MODULO p^m

Raf Cluckers

Tome 135
Fascicule 4

2007

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 1-

Bull. Soc. math. France
135 (4), 2007, p. 475–

THE FUNDAMENTAL THEOREM OF PREHOMOGENEOUS VECTOR SPACES MODULO p^m (WITH AN APPENDIX BY F. SATO)

by Raf Cluckers & Adriaan Herremans

Abstract. — For a number field K with ring of integers $\mathcal{O}_K$, we prove an analogue over finite rings of the form $\mathcal{O}_K/\mathcal{P}^m$ of the fundamental theorem on the Fourier transform of a relative invariant of prehomogeneous vector spaces, where $\mathcal{P}$ is a big enough prime ideal of $\mathcal{O}_K$ and $m > 1$. In the appendix, F. Sato gives an application of the Theorems 1.1, 1.3 and the Theorems A, B, C in J. Denef and A. Gyoja [*Character sums associated to prehomogeneous vector spaces*, Compos. Math., **113** (1998), 237–346] to the functional equation of L-functions of Dirichlet type associated with prehomogeneous vector spaces.

Texte recu le 12 septembre 2006, révisé le 16 novembre 2006

Raf Cluckers, Katholieke Universiteit Leuven, Departement wiskunde, Celestijnenlaan 200B, B-3001 Leuven (Belgium). Current address: École Normale Supérieure, Département de mathématiques et applications, 45 rue d'Ulm, 75230 Pari Cedex 05 (France)
E-mail : cluckers@ens.fr • *Url :* www.dma.ens.fr/~cluckers/
Adriaan Herremans, Katholieke Universiteit Leuven, Departement wiskunde, Celestijnenlaan 200B, B-3001 Leuven (Belgium)
E-mail : adriaan.herremans@wis.kuleuven.ac.be

2000 Mathematics Subject Classification. — 11S90, 11L07, 11M41, 11T24, 11L05, 20G40.

Key words and phrases. — Prehomogeneous vector spaces, L-functions, Bernstein-Sato polynomial, fundamental theorem of prehomogeneous vector spaces, exponential sums.

During the preparation of the paper, the authors were postdoctoral fellows of the Fund for Scientific Research – Flanders (Belgium) (F.W.O.). The first author was partially supported by The European Commission - Marie Curie European Individual Fellowship with contract number HPMF CT 2005-007121.

Résumé (*Théorème fondamental des espaces vectoriels préhomogènes modulo p^m. Avec un appendice par F. Sato*)

Soit K un corps de nombres avec anneaux d'entiers $\mathcal{O}_K$; nous prouvons un analogue, sur des anneaux finis de la forme $\mathcal{O}_K/\mathcal{P}^m$, du théorème fondamental sur la transformation de Fourier de l'invariante relative d'un espace vectoriel préhomogène. Ici, $\mathcal{P}$ est un idéal premier assez grand de $\mathcal{O}_K$ et $m > 1$. Dans l'appendice, F. Sato donne une application des théorèmes 1.1, 1.3 et des théorèmes A, B, C de J. Denef et A. Gyoja [*Character sums associated to prehomogeneous vector spaces*, Compos. Math., **113** (1998), 237–346] à l'équation fonctionelle de L-fonctions de type Dirichlet associées aux espaces vectorielles préhomogènes.

1. Introduction

We prove an analogue over finite rings of the fundamental theorem on the Fourier transform of a relative invariant of prehomogeneous vector spaces. In general, this fundamental theorem expresses the Fourier transform of $\chi(f)$, with χ a multiplicative (quasi-)character and f a relative invariant, in terms of $\chi(f^\vee)^{-1}$, with $f^\vee$ the dual relative invariant. Roughly speaking, M. Sato [18] proved the fundamental theorem over archimedian local fields, J. Igusa [7] over p-adic number fields, and J. Denef and A. Gyoja [5] over finite fields of big enough characteristic. In [9], the regular finite field case is reproved. When the prehomogeneous vector space is regular and defined over a number field K we prove an analogue of the fundamental theorem over rings of the form $\mathcal{O}_K/\mathcal{P}^m$, where $\mathcal{P}$ is a big enough prime ideal of the ring of integers $\mathcal{O}_K$ of K and $m > 1$, see Theorem 1.1. This result is derived from the results of [5] by using explicit calculations of exponential sums over the rings $\mathcal{O}_K/\mathcal{P}^m$.

In [16], F. Sato introduces L-functions of Dirichlet type associated to regular prehomogeneous vector spaces. In the appendix by F. Sato to this paper, our results are used to obtain functional equations for these L-functions and, under extra conditions, their entireness.

To state the main results, we fix our notation on prehomogeneous vector spaces. Let (G, ρ, V) be a reductive prehomogeneous vector space, meaning that G is a connected complex linear reductive algebraic group, $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional rational representation, and V has an open G-orbit which is denoted by Ω. Assume that (G, ρ, V) has a relative invariant $0 \neq f \in \mathbb{C}[V]$ with character $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$, that is, $f(gv) = \phi(g)f(v)$ for all $g \in G$ and $v \in V$. We assume that f is a *regular* relative invariant, namely, $\Omega = V \setminus f^{-1}(0)$ is a single G-orbit. Writing $\rho^\vee : G \to \mathrm{GL}(V^\vee)$ for the dual of ρ, $(G, \rho^\vee, V^\vee)$ is also a prehomogeneous vector space, with an open G-orbit which is denoted by $\Omega^\vee$, and there exists a relative invariant $0 \neq f^\vee \in \mathbb{C}[V^\vee]$ whose character is ϕ^{-1}. Then $\Omega^\vee = V^\vee \setminus f^{\vee -1}(0)$. The map $F := \mathrm{grad} \log f$ is an isomorphism between Ω and $\Omega^\vee$ with inverse $F^\vee := \operatorname{grad}\log f^\vee$. One has $\dim V = \dim V^\vee =: n$ and $\deg f = \deg f^\vee =: d$.

Let K be a number field with ring of integers $\mathcal{O}_K$. Suppose that (G, ρ, V) is defined over K. We fix a basis of the K-vector space $V(K)$ and we suppose that f is in $K[V]$ and has coefficients in $\mathcal{O}_K$ (with respect to the fixed K-basis of $V(K)$). Similarly we suppose that $f^\vee$ is in $K[V^\vee]$ and has coefficients in $\mathcal{O}_K$ (with respect to the basis of $V^\vee$ dual to the fixed basis of V). Write $V(\mathcal{O}_K)$ for the points of $V(K)$ with coefficients in $\mathcal{O}_K$ (with respect to the fixed K-basis of $V(K)$), and similarly for $V^\vee(\mathcal{O}_K)$. For I an ideal of $\mathcal{O}$, write $V(\mathcal{O}_K/I)$ for the reduction modulo I of the lattice $V(\mathcal{O}_K)$.

The Bernstein-Sato polynomial $b(s)$ of f is defined by

$$f^\vee(\operatorname{grad}_x) f(x)^{s+1} = b(s) f(x)^s.$$

Write b_0 for the coefficient of the term of highest degree of $b(s)$; one has $b_0 \in K$.

The following theorem is an analogue of the fundamental theorem for prehomogeneous vector spaces.

THEOREM 1.1. — *Let $m \geq 2$ be an integer, $\mathcal{P}$ be a prime ideal of $\mathcal{O}_K$ above a big enough prime $p \in \mathbb{Z}$, χ be a primitive multiplicative character modulo $\mathcal{P}^m$ (extended by zero outside the multiplicative units), and let ψ be a primitive additive character modulo $\mathcal{P}^m$. Write $q := \#(\mathcal{O}_K/\mathcal{P})$. For $L \in V^\vee(\mathcal{O}_K/\mathcal{P}^m)$ write*

$$S(L) := \sum_{x \in V(\mathcal{O}_K/\mathcal{P}^m)} \chi\big(f(x)\big)\psi\big(L(x)\big).$$

Then the following hold:

1) *if $f^\vee(L) \not\equiv 0 \bmod \mathcal{P}$, then*

$$S(L) = q^{\frac{1}{2}mn}\Big(\frac{\sum_{y\in\mathcal{O}_K/\mathcal{P}^m} \chi^d(y)\psi(y)}{q^{\frac{1}{2}m}}\Big)\chi\Big(\frac{b_0 f^\vee(L)^{-1}}{d^d}\Big)\alpha(\chi, m)^{n-1}\kappa^\vee(L),$$

where $\kappa^\vee(L)$ and $\alpha(\chi, m)$ are 1 or -1;

2) *if $f^\vee(L) \equiv 0 \bmod \mathcal{P}$, then $S(L) = 0$.*

The essential (and typical) content of this fundamental theorem is that the discrete Fourier transform of the function $\chi(f)$ on $V(\mathcal{O}_K/\mathcal{P}^m)$ is equal to the function $\chi(f^\vee)^{-1}$ on $V^\vee(\mathcal{O}_K/\mathcal{P}^m)$ times some factors, and vice versa.

The first part of Theorem 1.1 is obtained by combining explicit calculations of character sums of quadratic functions (§2) and of discrete Fourier transforms (§4), a p-adic version of Morse's lemma (§3), and results of [5]. The second part of Theorem 1.1 is established by comparing the L_2-norms of $\chi(f)$ and of its discrete Fourier transform.

We also obtain explicit formulas for the constants $\kappa^\vee(L)$ and $\alpha(\chi,m)$ of Theorem 1.1, by using work in [5] and elementary calculations. To state these formulas we use the notion of the discriminant of a matrix, as in [5, 9.1.0].

Definition 1.2. — For a symmetric (n,n)-matrix A with entries in a field k, if ${}^tXAX = \operatorname{diag}(a_1,\dots,a_m,0,\dots,0)$ with $X \in \mathrm{GL}_n(k)$, ${}^tX$ its transposed, and $a_i \in k^\times$, put

$$\Delta(A) := \prod_{i=1}^{m} a_i \in k^\times/k^{\times 2},$$

with $k^{\times 2}$ the squares in $k^\times$, and call it the *discriminant* of A.

Write $k_{\mathcal{P}}$ for the finite field $\mathcal{O}_K/\mathcal{P}$ and $k_{\mathcal{P}}^{\times 2}$ for the squares in $k_{\mathcal{P}}^\times$. For $m > 1$ and L in $V^\vee(\mathcal{O}_K/\mathcal{P}^m)$ with $f^\vee(L) \not\equiv 0 \bmod \mathcal{P}$, denote by $h^\vee(L)$ the image in $k_{\mathcal{P}}^\times/k_{\mathcal{P}}^{\times 2}$ of the discriminant of the matrix $(\partial^2 \log f^\vee(L)/\partial y_i \partial y_j)_{ij}$, where $\{y_1,\dots,y_n\}$ is the previously fixed K-basis of $V^\vee(K)$. Write $\chi_{\frac{1}{2}}$ for the Legendre symbol mod $\mathcal{P}$. We then obtain

Theorem 1.3. — *The following hold in case* 1) *of Theorem* 1.1:

1) $\kappa^\vee(L) = \chi_{\frac{1}{2}}(-d\,2^{n-1}\,h^\vee(L))^m$;
2) $\alpha(\chi,m) = 1$ *for m even*;
3) $\alpha(\chi,m) = G(\chi_{\frac{1}{2}},\psi')/\sqrt{q}$ *for m odd, with ψ' any additive character defined by $y \mapsto \chi(1+\pi_{\mathcal{P}}^{m-1}y)$, $\pi_{\mathcal{P}}$ any element in $\mathcal{P}$ of $\mathcal{P}$-adic order* 1, *$\chi_{\frac{1}{2}}$ the Legendre symbol* mod $\mathcal{P}$, *and $G(.,.)$ the classical Gauss sum.*

Remark 1.4. — It is interesting to compare the formulas of Theorems 1.1 and 1.3 to the formulas for $m = 1$ given in [5]; it seems that for $m = 1$ the formulas depend more on subtle information of the Bernstein-Sato polynomial of f.

Acknowledgement. — The authors would like to express gratitude to Jan Denef and Akihiko Gyoja for many helpful comments during the preparation of this paper. We are indebted to Jan Denef for giving the essential outline of the proofs of the main results, and to Fumihiro Sato for agreeing on writing an appendix to this paper.