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# APPROXIMATELY EINSTEIN ACH METRICS, VOLUME RENORMALIZATION, AND AN INVARIANT FOR CONTACT MANIFOLDS

by Neil Seshadri

Abstract. — To any smooth compact manifold $M$ endowed with a contact structure $H$ and partially integrable almost CR structure $J$, we prove the existence and uniqueness, modulo high-order error terms and diffeomorphism action, of an approximately Einstein ACH (asymptotically complex hyperbolic) metric $g$ on $M \times (-1, 0)$.

We consider the asymptotic expansion, in powers of a special defining function, of the volume of $M \times (-1, 0)$ with respect to $g$ and prove that the log term coefficient is independent of $J$ (and any choice of contact form $\theta$), i.e., is an invariant of the contact structure $H$.

The approximately Einstein ACH metric $g$ is a generalisation of, and exhibits similar asymptotic boundary behaviour to, Fefferman's approximately Einstein complete Kähler metric $g_+$ on strictly pseudoconvex domains. The present work demonstrates that the CR-invariant log term coefficient in the asymptotic volume expansion of $g_+$ is in fact a contact invariant. We discuss some implications this may have for CR $Q$-curvature.

The formal power series method of finding $g$ is obstructed at finite order. We show that part of this obstruction is given as a one-form on $H^*$. This is a new result peculiar to the partially integrable setting.

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Résumé (*Métriques presque d'Einstein ACH, renormalisation de volume, et un invariant pour les variétés de contact*)

Pour toute variété lisse compacte $M$ munie d'une structure de contact $H$ et d'une structure presque CR partiellement intégrable $J$, nous démontrons l'existence et l'unicité, à des termes d'erreur de degré supérieur et action de difféomorphisme près, d'une métrique presque d'Einstein ACH (asymptotiquement complexe hyperbolique) $g$ sur $M \times (-1, 0)$.

Nous considérons le développement asymptotique, en des puissances d'une fonction définissante spéciale, du volume de $M \times (-1, 0)$ par rapport à $g$. Nous démontrons que le coefficient du terme logarithmique est indépendant de $J$ (et du choix de la forme de contact $\theta$) ; par conséquent, c'est un invariant de la structure de contact $H$.

La métrique presque d'Einstein ACH $g$ est une généralisation de la métrique presque d'Einstein kählérienne complète $g_+$ de Fefferman sur les domaines strictement pseudoconvexes. Elle a également un comportement asymptotique similaire au bord. Le présent travail démontre que le coefficient du terme logarithmique CR-invariant dans le développement asymptotique du volume de $g_+$ est, en fait, un invariant de contact. Nous traitons également quelques implications possibles pour la $Q$-courbure CR.

La méthode de trouver $g$ par le biais de séries formelles comporte une obstruction d'ordre fini. Nous démontrons que cette obstruction est partiellement donnée par une 1-forme sur $H^*$. Ceci est un résultat nouveau particulier au contexte partiellement intégrable.

## 1. Introduction

In a previous paper [29], inspired by Graham [19], we studied volume renormalization for Fefferman's approximately Einstein complete Kähler metric on a strictly pseudoconvex domain in a complex manifold. We considered the asymptotic expansion, in powers of a special boundary defining function, for the volume of this domain and showed that the coefficient $L$ of the log term in this expansion is an invariant of the boundary CR structure. In complex dimension two $L$ always vanishes. In higher dimensions, we showed in subsequent work [30] that $L$ is moreover invariant under (integrable) deformations of the CR structure. This led us to speculate that $L$ is in fact an invariant of the *contact* structure on the boundary. One of the purposes of the present paper is to show that this is indeed the case.

To make sense of the last statement, in this paper we first generalise $L$ to be defined on an aribitrary smooth compact orientable contact manifold $(M, H)$. We do so by first endowing $(M, H)$ with a partially integrable almost CR structure $J$, a generalisation of an integrable CR structure. (Background material with definitions will follow in the next section.) We then define $L$ in this generalised setting as the log term coefficient in the volume expansion of an approximately Einstein ACH (asymptotically complex hyperbolic) metric $g$ on $X := M \times (-1, 0)$. Our definition of ACH is contained is Definition 2.1;

suffice for now to say that such a metric exhibits similar boundary asymptotics to those of Fefferman's approximately Einstein complete Kähler metric. The special defining function $\varphi$ used for the expansion is from Lemma 2.2 and corresponds to a choice of contact form $\theta$ for $(M, H)$. Then we have:

THEOREM 4.4. — *Define the* Einstein tensor *by* $\mathrm{Ein} := \mathrm{Ric} + 2(n+2)g$. *Then there exists an ACH metric $g$ on $X$ that solves* $\mathrm{Ein} = O(\varphi^n)$ *with* $\mathrm{Ein}(W, Z) = O(\varphi^{n+1})$ *for $W \in H$ and $Z \in TM$. Moreover if $g'$ is another such ACH metric then there exists a diffeomorphism $F$ of $\overline{X}$ that restricts to the identity on $M$ with $g' = F^*g + \varphi^n G$, where $G$ is O(1) and $G(W, Z) = O(\varphi)$ for $W \in H$ and $Z \in TM$.*

There are two key initial steps in the proof of Theorem 4.4, both reminiscent of similar steps in the integrable CR setting [29]. The first is to make a special choice of coframe for $T\overline{X}$ to allow $g$ to be written in a normal form—see §2. The second is to write the Levi-Civita connection and curvature of $g$ in terms of local data associated with (the extension to $\overline{X}$ of) a canonical connection adapted to $(M, H, J, \theta)$. We use a connection introduced by Tanno [32] and call it the TWT connection, since it reduces to the more familiar Tanaka–Webster connection when $J$ is integrable. Details about the TWT connection are in §3.

The remainder of the proof of Theorem 4.4 uses methods from Graham–Hirachi [20]. We solve the Einstein equation iteratively to determine $g$ up to a finite order and then use the contracted Bianchi identity to prove that all the components of Ein vanish to the correct order—see §4.

The main result of this paper is the following:

MAIN THEOREM. — *The log term coefficient $L$ in the asymptotic volume expansion of $X$ with respect to an approximately Einstein ACH metric $g$ is an invariant of the contact structure $H$ on $M$.*

Since any contact manifold admits a contractible homotopy class of partially integrable almost CR structures, the proof of the Main Theorem follows from a deformation argument, similar to that in [20] and [30]—see §5.

Now when the partially integrable almost CR structure $J$ is integrable, so that $(M, H, J)$ is a CR manifold, the approximately Einstein ACH metric from Theorem 4.4 does in fact coincide (modulo high order error terms and diffeomorphism action) as a Riemannian metric with Fefferman's approximately Einstein complete Kähler metric—see §6. The respective special defining functions also coincide, hence the log term coefficients $L$ coming from the two volume renormalization procedures (i.e., in this paper and [29]) agree.

When $M$ has dimension 3, $J$ is automatically integrable. In this dimension $L$ always vanishes ([23], [29]). Whether there exist nonzero $L$ in higher dimensions is an open question. Direct calculation using our volume renormalization techniques seems a computationally infeasible task. In the integrable CR setting the fact that $L$ is a constant multiple of the integral of CR $Q$-curvature makes settling the question of its (non)vanishing an even more pertinent task. The contact-invariance of $L$ proved in this paper could be a useful contribution to a solution to this problem. We speculate more on this matter and briefly discuss some other recently disovered contact invariants in the final §7.

Some remarks are in order about the literature on ACH metrics. Our definition of ACH is closest to that of Guillarmou–Sá Barreto [22], which in turn is based on the formalism of so-called Θ-metrics from Epstein–Melrose–Mendoza [9]. ACH-like metrics have also been studied by Roth [27], Biquard [2], Biquard–Herzlich [3] and Biquard–Rollin [5]; these authors also considered Einstein conditions, although with different purposes in mind to ours.

Finally let us make some comments about *ACHE (ACH Einstein) metrics*, by which we mean ACH metrics satisfying $\mathrm{Ein} = O(\varphi^m)$, for all $m$. With additional smoothness restraints, the existence of such metrics is in general obstructed by certain tensors. In §4 we define the obstruction tensors (for $T$ the Reeb field and $W_A$ in the contact direction)

$$\mathcal{B} := \varphi^{-n}\mathrm{Ein}(T,T)|_M,$$

$$\mathcal{O}_A := \varphi^{-(n+1)}\mathrm{Ein}(T,W_A)|_M,$$

and prove the following result.

Proposition 4.5. — (i) *The obstruction tensors $\mathcal{B}$ (a scalar function) and $\mathcal{O}_A$ are well-defined independently of the ambiguity in approximately Einstein ACH metric $g$.*
(ii) *Under a change in contact form $\widehat{\theta} = e^{2\Upsilon}\theta$, the obstruction tensors satisfy*

$$\widehat{\mathcal{B}} = e^{-2(n+2)\Upsilon}\mathcal{B}$$

*and*

$$\widehat{\mathcal{O}}_A = e^{-2(n+2)\Upsilon}(\mathcal{O}_A - 2i\varphi^{-(n+1)}\mathrm{Ein}(\Upsilon^\alpha W_\alpha - \Upsilon^{\overline{\beta}}W_{\overline{\beta}}, W_A)|_M).$$

(iii) *If $(M,H,J)$ is such that $\mathcal{B}$ vanishes then, under a change in contact form $\widehat{\theta} = e^{2\Upsilon}\theta$, the obstruction $\mathcal{O}_A$ satisfies*

$$\widehat{\mathcal{O}}_A = e^{-2(n+2)\Upsilon}\mathcal{O}_A.$$

That there exists a secondary obstruction $\mathcal{O}_A$ given as a one-form in $H^*$ is a novel feature of this partially integrable setting, since in the integrable case it is well-known ([11], [24], [18]) that the only obstruction to appear is a scalar function. Studying further the obstruction tensors and in particular their relation with the (almost) CR deformation complex should be interesting (cf. [17] in the setting of conformal geometry).