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ON THE IWAHORI WEYL GROUP

BY TIMO RICHARZ

ABSTRACT. — Let F be a discretely valued complete field with valuation ring \mathcal{O}_F and perfect residue field k of cohomological dimension ≤ 1 . In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F -groups to the case of arbitrary connected reductive F -groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4].

Let F be a discretely valued complete field with valuation ring \mathcal{O}_F and perfect residue field k of cohomological dimension ≤ 1 . In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected F -groups to the case of arbitrary connected reductive F -groups. If k is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4]. Furthermore, we study the length function on the Iwahori-Weyl group, and use it to determine the number of points in a Bruhat cell, when k is a finite field. Except for Lemma 1.3 below, the results are independent of [4], and are directly based on the work of Bruhat and Tits [2], [3].

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Let \bar{F} be the completion of a separable closure of F . Let \check{F} be the completion of the maximal unramified subextension with valuation ring $\mathcal{O}_{\check{F}}$ and residue field \bar{k} . Let $I = \text{Gal}(\bar{F}/\check{F})$ be the inertia group of \check{F} , and let $\Sigma = \text{Gal}(\check{F}/F)$.

Let G be a connected reductive group over F , and denote by $\mathcal{B} = \mathcal{B}(G, F)$ the enlarged Bruhat-Tits building. Fix a maximal F -split torus A . Let $\mathcal{A} = \mathcal{A}(G, A, F)$ be the apartment of \mathcal{B} corresponding to A .

1.1. Definition of the Iwahori-Weyl group. — Let $M = Z_G(A)$ be the centralizer of A , an anisotropic group, and let $N = N_G(A)$ be the normalizer of A . Denote by $W_0 = N(F)/M(F)$ the relative Weyl group.

DEFINITION 1.1. — i) The *Iwahori-Weyl group* $W = W(G, A, F)$ is the group

$$W \stackrel{\text{def}}{=} N(F)/M_1,$$

where M_1 is the unique parahoric subgroup of $M(F)$.

ii) Let $\mathfrak{a} \subset \mathcal{A}$ be a facet and $P_{\mathfrak{a}}$ the associated parahoric subgroup. The subgroup $W_{\mathfrak{a}}$ of the Iwahori-Weyl group corresponding to \mathfrak{a} is the group

$$W_{\mathfrak{a}} \stackrel{\text{def}}{=} P_{\mathfrak{a}} \cap N(F)/M_1.$$

The group $N(F)$ operates on \mathcal{A} by affine transformations

$$(1.1) \quad \nu : N(F) \longrightarrow \text{Aff}(\mathcal{A}).$$

The kernel $\ker(\nu)$ is the unique maximal compact subgroup of $M(F)$ and contains the compact group M_1 . Hence, the morphism (1.1) induces an action of W on \mathcal{A} .

Let G_1 be the subgroup of $G(F)$ generated by all parahoric subgroups, and define $N_1 = G_1 \cap N(F)$. Fix an alcove $\mathfrak{a}_C \subset \mathcal{A}$, and denote by B the associated Iwahori subgroup. Let \mathbb{S} be the set of simple reflections at the walls of \mathfrak{a}_C . By Bruhat and Tits [3, Prop. 5.2.12], the quadruple

$$(1.2) \quad (G_1, B, N_1, \mathbb{S})$$

is a (double) Tits system with affine Weyl group $W_{\text{af}} = N_1/N_1 \cap B$, and the inclusion $G_1 \subset G(K)$ is B - N -adapted of connected type.

LEMMA 1.2. — i) *There is an equality $N_1 \cap B = M_1$.*

ii) *The inclusion $N(F) \subset G(F)$ induces a group isomorphism $N(F)/N_1 \xrightarrow{\cong} G(F)/G_1$.*

Proof. — The group $N_1 \cap B$ operates trivially on \mathcal{A} and so is contained in $\ker(\nu) \subset M(F)$. In particular, $N_1 \cap B = M(F) \cap B$. But $M(F) \cap B$ is a parahoric subgroup of $M(F)$ and therefore equal to M_1 .

The group morphism $N(F)/N_1 \rightarrow G(F)/G_1$ is injective by definition. We have to show that $G(F) = N(F) \cdot G_1$. This follows from the fact that the inclusion $G_1 \subset G(F)$ is B - N -adapted, cf. [2, 4.1.2]. □

Kottwitz defines in [5, §7] a surjective group morphism

$$(1.3) \quad \kappa_G : G(F) \longrightarrow X^*(Z(\hat{G})^I)^\Sigma.$$

Note that in [*loc. cit.*] the residue field k is assumed to be finite, but the arguments extend to the general case.

LEMMA 1.3. — *There is an equality $G_1 = \ker(\kappa_G)$ as subgroups of $G(F)$.*

Proof. — For any facet \mathfrak{a} , let $\text{Fix}(\mathfrak{a})$ be the subgroup of $G(F)$ which fixes \mathfrak{a} pointwise. The intersection $\text{Fix}(\mathfrak{a}) \cap \ker(\kappa_G)$ is the parahoric subgroup associated to \mathfrak{a} , cf. [4, Proposition 3]. This implies $G_1 \subset \ker(\kappa_G)$. For any facet \mathfrak{a} , let $\text{Stab}(\mathfrak{a})$ be the subgroup of $G(F)$ which stabilizes \mathfrak{a} . Fix an alcove \mathfrak{a}_C . There is an equality

$$(1.4) \quad \text{Fix}(\mathfrak{a}_C) \cap G_1 = \text{Stab}(\mathfrak{a}_C) \cap G_1,$$

and (1.4) holds with G_1 replaced by $\ker(\kappa_G)$, cf. [4, Lemma 17]. Assume that the inclusion $G_1 \subset \ker(\kappa_G)$ is strict, and let $\tau \in \ker(\kappa_G) \setminus G_1$. By Lemma 1.2 ii), there exists $g \in G_1$ such that $\tau' = \tau \cdot g$ stabilizes \mathfrak{a}_C , and hence τ' is an element of the Iwahori subgroup $\text{Stab}(\mathfrak{a}_C) \cap \ker(\kappa_G)$. This is a contradiction, and proves the lemma. □

By Lemma 1.2, there is an exact sequence

$$(1.5) \quad 1 \longrightarrow W_{\text{af}} \longrightarrow W \xrightarrow{\kappa_G} X^*(Z(\hat{G})^I)^\Sigma \longrightarrow 1.$$

The stabilizer of the alcove \mathfrak{a}_C in W maps isomorphically onto $X^*(Z(\hat{G})^I)^\Sigma$ and presents W as a semidirect product

$$(1.6) \quad W = X^*(Z(\hat{G})^I)^\Sigma \ltimes W_{\text{af}}.$$

For a facet \mathfrak{a} contained in the closure of \mathfrak{a}_C , the group $W_{\mathfrak{a}}$ is the parabolic subgroup of W_{af} generated by the reflections at the walls of \mathfrak{a}_C which contain \mathfrak{a} .

THEOREM 1.4. — *Let \mathfrak{a} (resp. \mathfrak{a}') be a facet contained in the closure of \mathfrak{a}_C , and let $P_{\mathfrak{a}}$ (resp. $P_{\mathfrak{a}'}$) be the associated parahoric subgroup. There is a bijection*

$$\begin{aligned} W_{\mathfrak{a}} \backslash W / W_{\mathfrak{a}'} &\xrightarrow{\cong} P_{\mathfrak{a}} \backslash G(F) / P_{\mathfrak{a}'} \\ W_{\mathfrak{a}} w W_{\mathfrak{a}'} &\longmapsto P_{\mathfrak{a}} n_w P_{\mathfrak{a}'}, \end{aligned}$$

where n_w denotes a representative of w in $N(F)$.