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THE SPECTRUM OF A SCHRÖDINGER OPERATOR IN A WIRE-LIKE DOMAIN WITH A PURELY IMAGINARY DEGENERATE POTENTIAL IN THE SEMICLASSICAL LIMIT

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THE SPECTRUM OF A SCHRÖDINGER OPERATOR IN A WIRE-LIKE DOMAIN WITH A PURELY IMAGINARY DEGENERATE POTENTIAL IN THE SEMICLASSICAL LIMIT

Y. Almog, B. Helffer

Abstract. – Consider a two-dimensional domain shaped like a wire, not necessarily of uniform cross section. Let V denote an electric potential driven by a voltage drop between the conducting surfaces of the wire. We consider the operator $\mathscr{A}_h = -h^2\Delta + iV$ in the semi-classical limit $h \to 0$. We obtain both the asymptotic behavior of the left margin of the spectrum, as well as resolvent estimates on the left side of this margin. We extend here previous results obtained for potentials for which the set where the current (or ∇V) is normal to the boundary is discrete, in contrast with the present case where V is constant along the conducting surfaces.

Résumé **(Le spectre d'un opérateur de Schrödinger à potentiel purement imaginaire et dégénéré dans un domaine filaire)**

Nous considérons un domaine filaire sans supposer qu'il a une section uniforme. Pour un potentiel électrique V créé par une différence de tension entre les deux surfaces conductrices, nous considérons l'opérateur $\mathscr{A}_h = -h^2\Delta + iV$ dans la limite semi-classique $h \to 0$. Nous obtenons le comportement asymptotique du bas de la partie réelle de son spectre de même que des estimations de sa résolvante en dessous de ce seuil. Nous étendons les résultats obtenus précédemment dans le cas où le gradient du potentiel n'est normal à la frontière qu'en un nombre fini de points en contraste au cas présent où V est constant sur les surfaces conductrices.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Main assumptions

We consider the operator

$$\mathscr{A}_h = -h^2\Delta + iV, \tag{1.1a}$$

defined on

$$D(\mathscr{A}_h) = \{u \in H^2(\Omega,) | u|_{\partial\Omega_D} = 0; \partial u/\partial\nu|_{\partial\Omega_N} = 0\}. \tag{1.1b}$$

In the above, $\Omega \subset \mathbb{R}^2$ denotes a bounded, simply connected domain which has the same characteristics as in [**5, 7**]. In particular its boundary $\partial\Omega$ contains two disjoint open subsets $\partial\Omega_D$ and $\partial\Omega_N$ such that

$$\overline{\partial\Omega_D} \cup \overline{\partial\Omega_N} = \partial\Omega,$$

where $\partial\Omega_D$ is a union of two disjoint smooth interfaces on which we prescribe a Dirichlet boundary condition, and $\partial\Omega_N$ is a union of two disjoint smooth interfaces on which we prescribe a Neumann boundary condition. Hence $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ consists of four points which will be called corners. The analysis can be extended to domains, where $\partial\Omega_D$ (and $\partial\Omega_N$) consists of a greater number of disjoint components. In the interest of simplicity we shall confine ourselves to the simplest possible case.

In the context of superconductivity we may say that $\partial\Omega_D$ and $\partial\Omega_N$, are respectively adjacent either to a normal metal or to an insulator. We denote each connected component of $\partial\Omega_\#$ ($\# \in \{D, N\}$) by a superscript $i \in \{1, 2\}$, i.e.,

$$\partial\Omega_\# = \partial\Omega^1_\# \cup \partial\Omega^2_\#, \quad \# \in \{D, N\}.$$

We say that $\partial\Omega$ is of class $C^{n,+}$ for some $n \in \mathbb{N}$, if there exists $\check{\beta} > 0$ such that $\partial\Omega$ is of class $C^{n,\check{\beta}}$. As in [**3, 5, 7**] we make the following assumptions on $\partial\Omega$

$$(R1) \begin{cases} \text{(a) } \overline{\partial\Omega_\#} \text{ is of class } C^{n,+} \text{ for } \# \in \{D, N\}; \\ \text{(b) near each corner, } \overline{\partial\Omega_D} \text{ and } \overline{\partial\Omega_N} \text{ meet with an angle of } \frac{\pi}{2}. \end{cases} \tag{1.2}$$

We define n for each result separately (but always have $n \geq 2$). We occasionally use the notation $(R1(n))$ to specify n in the assumption.

Near the corners, we assume in addition that there exists a smooth transformation, mapping the vicinity of the corner onto a vicinity of rectangular corner. More precisely

$$(R2)\begin{cases} \text{For each corner } \mathfrak{c}, \text{ there exist } R>0 \text{ and an invertible holomorphic function} \\ \Phi \text{ in } B(\mathfrak{c},R)\cap\Omega, \text{ which is in addition in } C^{n,+}(\bar\Omega\cap B(\mathfrak{c},R)), \\ \text{such that } \Phi(\mathfrak{c})=0, \Phi(B(\mathfrak{c},R)\cap\Omega)\subset Q:=\mathbb{R}_+\times\mathbb{R}_+, \\ \text{and } \Phi(\partial\Omega\cap B(\mathfrak{c},R))\subset(\mathbb{R}_+\times\{0\})\cup(\{0\}\times\mathbb{R}_+)\cup\{0\}. \end{cases} \tag{1.3}$$

Again we use $(R2(n))$ to specify n in the assumption.

We consider potentials $V\in H^2(\Omega)$ satisfying

$$\begin{cases} \Delta V=0 & \text{in } \Omega, \\ V=C_i & \text{on } \partial\Omega_D^i \text{ for } i=1,2, \\ \frac{\partial V}{\partial\nu}=0 & \text{on } \partial\Omega_N, \end{cases} \tag{1.4}$$

describing a potential drop along a wire.

Assumptions $(R1(n))$ and $(R2(n))$ imply that $V\in C^{n,+}(\bar\Omega)$. Away from the corners, we may rely on Schauder estimates to establish the desired regularity. In the neighborhood of a corner, we may use the conformal map given by Assumption $(R2(n))$ to obtain a problem for V in a right-angled sector. Then we can use a reflection argument to establish the announced regularity of V (cf. **[3, 5]** for instance).

We assume further, as in **[4]**, that V satisfies

$$|\nabla V(x)|\neq 0, \forall x\in\overline{\Omega}. \tag{1.5}$$

This implies that

$$C_1\neq C_2.$$

We can indeed follow one component of $\partial\Omega_N$ between two corners and observe that the tangential derivative of V never vanishes (cf. **[3]**).

Equation (1.4) has extensively been studied in the literature. We refer to **[18]**, where explicitly known solutions, for many simple domains including the square, are listed. Figure 1 presents a typical sample with properties (R1) and (R2), where the current flows into the sample from one connected component of $\partial\Omega_D$, and exits from another part, disconnected from the first one. Most wires would fall into the above class of domains.

Note that, V being constant on each connected component of $\partial\Omega_D$, we have

$$|\nabla V|=|\partial V/\partial\nu| \text{ on } \partial\Omega_D.$$

We distinguish in the sequel between two types of potentials satisfying (1.4).

V1 : Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}}|\partial V/\partial\nu|$ is attained, lie in $\partial\Omega_D$.

V2 : Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}}|\partial V/\partial\nu|$ is attained are corners.