Revue d'histoire des mathématiques,
8 (2002), p. 263–289.

NOTES & DÉBATS

IT'S NOT THAT THEY COULDN'T

Reviel Netz (*)

It's not that she couldn't,
It's not that she wouldn't,
And you know–it's not that she shouldn't:
It's just that she is
The laziest gal in town.

Cole Porter

Abstract. — The article offers a critique of the notion of 'concepts' in the history of mathematics. Authors in the field sometimes assume an argument from conceptual impossibility: that certain authors could not do X because they did not have concept Y. The case of the divide between Greek and modern mathematics is discussed in detail, showing that the argument from conceptual impossibility is empirically as well as theoretically flawed. An alternative account of historical diversity is offered, based on self-sustaining practices, as well as on divergence being understood not in terms of intellectual values themselves (which may well be universal) but in terms of their rankings within different cultures and epochs.

Résumé. — Ce n'est pas qu'ils n'auraient pas pu. — Cet article offre une critique de la notion de "concepts" en histoire des mathématiques. Certains historiens s'appuient parfois sur un argument mettant en avant une impossibilité conceptuelle, du style: certains auteurs ne pouvaient pas faire X, parce qu'ils n'avaient pas le concept Y. Nous discutons en détail ce que cela signifie dans le cas de la différence entre mathématiques grecques et mathématiques modernes. Nous montrons que l'argument de l'impossibilité conceptuelle est empiriquement et théoriquement peu solide. Pour rendre compte de la diversité historique, l'article offre une alternative fondée sur des pratiques qui s'auto-entretiennent et sur la notion de divergence interprétée non en termes des valeurs intellectuelles elles-mêmes (qui pourraient bien être universelles), mais des rangs que ces valeurs occupent dans différentes cultures et époques.

(*) Texte reçu le 11 septembre 2002, révisé le 4 mars 2003.
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Keywords: Ancient Greek Mathematics, methodology, Euclid, Archimedes, Hipparchus, Diophantus, Hero.

AMS classification: 01A20.

Is mathematics always the same? If not, why? Historians of mathematics keep returning to these fundamental questions. The very question of what is 'the same' in mathematics is not easy to answer. After all, mathematicians have always shown the surprising fact that things that appear different are truly–seen under the appropriate perspective–*the same*: not only in the twentieth century's hunt for isomorphism, but starting with such observations that the squares on the two sides of the right-angled triangle are, in some sense, *the same* as the square on the hypotenuse... [1] It is thus natural, faced with an alien piece of mathematics, to show that it is 'the same', in some mathematical sense, with a certain subset of contemporary mathematics. This gives rise to the following set of objects:

OPM– an Old Piece of Mathematics,

CPM– a Contemporary Piece of Mathematics (to which OPM is equivalent),

CM– the entirety of Contemporary Mathematics (of which CPM is no more than a subset).

At this stage, the historian who wishes to say that mathematics is not always the same has two related routes open. One is to argue that while, from a certain perspective, OPM and CPM are indeed equivalent, they are also different enough from each other to merit the label 'different'.[2] This however seems weak on its own: no one ever denied the difference, but the question is, why should this difference matter once the basic equivalence is perceived? After all–is this not a mere matter of notation? Hence the second route: to argue that CPM is a subset of CM for a good reason: the way in which the mathematics of OPM was done made it impossible to do any mathematics but OPM, and so the modern equivalent to OPM can be CPM alone, and not CM as a whole. Mathematics is not always the same because, at different periods, different kinds of mathematics were possible. Transforming OPM into its contemporary equivalent, CPM, is

[1] [Goldstein 1995] is a fundamental study of 'the same' in mathematics, dedicated to the question: when are different mathematical proofs and propositions 'the same'?

[2] It is in fact difficult to define the 'equivalence' operative in this case. The standard example–the equivalence of Euclid's *Elements* II with algebraic equations–seems to suggest a meaning of 'equivalence' along the following lines: historians of mathematics often take two theorems to be equivalent when, from the perspective of the modern mathematician, the proof of any of the theorems serves to show, simultaneously, the truth of the other.

misleading: it obscures the idiosyncratic features of OPM that blocked it from becoming CM. No mere matter of notation, then: the difference between OPM and CPM is historically explanatory.

Such was the form of the most famous twentieth century debate in the historiography of mathematics. Unguru [1975] argued that Greek mathematics differs from its modern equivalents; Freudenthal [1977] and Weil [1978] had argued that this is a matter of notation only; Unguru wrapped up the discussion in Unguru [1979] with wide-ranging historiographical and indeed philosophical comments (more recently re-considered and expanded in [Fried and Unguru 2001]). At the heart of Unguru's reply–which has now become, to varying degrees, the established view in the community of historians of mathematics–lies the fundamental work by Jacob Klein [1934/1936, 1968], *Greek Mathematical Thought and the Origins of Algebra*. Klein's thesis was that Greek mathematics, for deep conceptual reasons, just could not become the same as modern mathematics, and must have had the form of dealing with the synthesis of isolated geometrical problems (instead of systematic algebraic analyses). Why? Because the Greeks did not possess the right kind of concepts: for algebra, one needs second-order concepts that refer to other concepts, but the Greeks had only first-order concepts, referring directly to reality. But let us leave aside the details of Klein's thesis and concentrate on the form of the argument. Klein's claim–the foundation of Unguru's critique–was that the difference in form between Greek mathematics and its modern counterpart was historically explanatory: to wit, it explained why Greek mathematics could not be modern. Why? Because modern mathematics, in the Greek context, was conceptually impossible.

Once again: my interest in this article is not in the detail of Klein's historical thesis.[3] I am interested in the form of the argument. I shall call this *the argument from conceptual impossibility*. Its shape is: 'for conceptual reasons, X could not do Y'. In an important recent article, '*Conceptual Divergence–Canons and Taboos–and Critique*: *Reflections on Explanatory Categories*', Jens Hoyrup [forthcoming] had challenged the very argument from conceptual impossibility. According to Hoyrup, we

[3] I have discussed Klein's thesis in detail in [Netz forthcominga], where I argue that the difference Klein had noticed–between a more 'isolated' and 'qualitative' approach in Greek mathematics as opposed to a more 'systematic' and 'quantitative' approach in modern mathematics can be explained in terms of changing mathematical practice.

are too hasty to speak of 'possibility' and 'impossibility', and we tend to draw the border between them too neatly. This article is written so as to support, qualify and I hope to complement Hoyrup's. In the first section I shall give several examples for what is typically taken to be the fundamental divide between ancient and modern mathematics: the more 'algebraical' or 'arithmetical' nature of modern mathematics. I shall show that *it's not that they couldn't*: Greek mathematicians could, and did on occasion, produce a more 'arithmetical' kind of mathematics. In the second section I shall consider together the examples from the first section, showing how, even absent the argument from conceptual impossibility, the difference between ancient and modern mathematics remains important. I shall also return to set out in more detail Hoyrup's account as well as my own, complementary historiographical approach.

1. THE NON-ARITHMETICAL CHARACTER OF GREEK MATHEMATICS

In what follows I draw upon several recent studies on Greek mathematics that, taken together, show the inadequacy of the argument from conceptual impossibility: wherever we look, we find exceptions to the rule of the non-arithmetical character of Greek mathematics. The moral, however, is not that we should give up the picture of Greek mathematics as non-arithmetical, but that we should give up the argument from conceptual impossibility.

What do we mean by the 'non-arithmetical character of Greek mathematics'? Several different things: arithmetical and numerical questions are less important than they are in other mathematical traditions; geometrical objects (which are the focus of interest) are understood in a non-quantitative way. Finally, the arithmetical system itself is patchy. It completely lacks the coherent structure of its modern counterpart, both in mathematical structure (where we have the well-understood logical sequence from integers through positive rationals and reals, and through negatives, to complex numbers) and in symbolism (where we use the decimal positional system). To the Greek, numbers are mysterious and clumsy to handle; to us, they are fully brought under the control of logic and are easy to deal with. Let us begin to note some exceptions to this picture.

1.1. Fractions

One central perceived difference between Greek and modern conceptions of number has to do with fractions. It has been argued in recent studies that the Greeks did not possess the concept of a common fraction, using instead either unit-fractions or ratios ([Knorr 1982], [Fowler 1992]). What we refer to as 'three over five', a numerical value, would be for them either unit fractions, that is, 'a half and a tenth' (a sum of numerical values) or a ratio, that is 'the ratio of three to five' (not a numerical value at all, but a relation). There is a mass of evidence where Greek mathematicians treat fractions in just this way−an evidence which seems to go beyond notational differences into mathematical practice itself: common fractions, unlike other representations of fractions, allow direct calculation with fractions of the form 'the nth of m multiplied by the qth of p gives nqth of mp'. (This direct calculation serves to put 'fractions' on a par with integers and in this way opens the way for the contemporary clear logical structure.)

I move on to discuss a new study of this question by Jean Christianidis [forthcoming]. Christianidis sets out from a quotation from David Fowler that is very relevant to our concerns:

"Just one example of some operation such as the addition, subtraction, multiplication, or division of two fractional quantities, expressed directly as something like 'the nth of m multiplied by the qth of p gives nqth of mp', and clearly *unrelated, by context, to any conception in terms of simple and compound parts*, could be fatal to my thesis that we have no good evidence for the Greek use or conception of common fractions. I know of no such example" [Fowler 1987, pp. 264–265].

Christianidis then observes that Diophantus' problem IV.36 contains just that. Not indeed in numerical terms, but in terms of Diophantus' 'syncopated algebra'. Still: Diophantus shows a clear sense of multiplication of fractions where the numerator is multiplied by the numerator and the denominator−by the denominator. Transcribing Diophantus' syncopated algebra into symbolic algebra, the essence of Christianidis' argument is that Diophantus, in IV.36, directly derives from the multiplication of '$(3x)/(x-3)$'[4] by '$(4x)/(x-4)$' the form '$(12x^2)/(x^2+12-7x)$'. While

[4] The original for '$(3x)/(x-3)$' was 'number, three, in the part of: number, one, lacking monads, three', or perhaps (depending on how syncopated Diophantus' original papyrus