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LÊ'S CONJECTURE FOR CYCLIC COVERS

by

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***Abstract.* —** We describe the link of the cyclic cover over a singularity of complex surface (S,p) totally branched over the zero locus of a germ of analytic function $(S,p) \to (\mathbf{C},0)$. As an application, we prove Lê's conjecture for this family of singularities *i.e.* that if the link is homeomorphic to the 3-sphere then the singularity is an equisingular family of unibranch curves.

***Résumé* (Conjecture de Lê pour les revêtements cycliques). —** Nous décrivons le « link » du revêtement cyclique sur une singularité de surface complexe (S,p) totalement ramifiée sur le lieu des zéros d'un germe de fonction analytique $(S,p) \to (\mathbf{C},0)$. A titre d'application, nous prouvons la conjecture de Lê pour cette famille de singularités, *i.e.* si le « link » est homéomorphe à la sphère de dimension 3, alors la singularité est une famille équisingulière de courbes unibranches.

1. Introduction

The topology of singularities of complex surfaces has been studied thoroughly in the case of isolated singularities (link, Milnor fibration, monodromy, etc.). For non isolated singularities the situation is less known and more mysterious.

By this work, we start a serie of papers devoted to the study of the link of a non isolated singularity (S,p) and its relations with the geometry of (S,p) through the resolution and with the analytic properties of (S,p).

If (S,p) is a singularity of surface, one denotes by $\mathcal{L}(S,p)$ its link. One of the first questions is to give a topological characterization of a non singular germ. When the singularity (S,p) is isolated, Mumford's theorem gives such a characterization in term of $\mathcal{L}(S,p)$, namely (S,p) is not singular if and only if the link $\mathcal{L}(S,p)$ is homeomorphic to the 3-sphere. If (S,p) is not isolated, this is not true. For instance if $(S,p) \subset (\mathbf{C}^3,0)$ is given by the equation $z^2 - x^3 = 0$, or more generally if (S,p) is an equisingular family of unibranch curves, then $\mathcal{L}(S,p)$ is also homeomorphic to $\mathbf{S}^3$.

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It has been conjectured by Lê D.T. (see for instance [**19**]) that the equisingular families of unibranch curves are the only cases in which $\mathcal{L}(S,p)$ is homeomorphic to $\mathbf{S}^3$. In this paper, we prove Lê's conjecture for the singularities obtained as the cyclic cover over a singularity of complex surface (S,p) totally branched over a curve (Theorem 5.1). The proof is based on the explicit description of the link of such a singularity by means of a plumbing graph which is the aim of Sections 2 to 4.

In Section 2, we study the topological action of the normalization morphism on the links of the singularity. Namely, if (S,p) is a singularity of surface, then the normalization morphism $n : \overline{S} \to S$ restricts to the links, providing a map $n_| : \mathcal{L}(\overline{S}) \to \mathcal{L}(S)$ which is an homeomorphism over the complementary of the singular locus L_{Σ_S} of S, and which is a cyclic cover over each connected component of L_{Σ_S}.

In Section 3, we present some definitions and results about Waldhausen multilinks and their fibrations over the circle which will be applied in the next sections to the Milnor fibrations of some germs of analytic functions $(S,p) \to (\mathbf{C},0)$ defined on a surface singularity (S,p).

In Section 4, we describe the link of any singularity of complex surface obtained as the cyclic cover over some germ of surface (S,p) totally ramified over a germ of curve. These singularities include for instance the germs of hypersurfaces in $(\mathbf{C}^3,0)$ with equations $f(x,y) - z^k = 0$ or $f(x,y) - z^k g(x,y) = 0$. Our method generalizes that developed in [**16**] for the singularities $f(x,y) - z^k = 0$ when f is reduced, using the theory of fibred Waldhausen multilinks developped in Section 3. Similar results have been obtained independently by A. Némethi and A. Szilárd ([**14**]) when (S,p) is normal by performing direct calculus on plumbing graphs.

The method is sumarized in algorithms 4.5 and 4.7. We give several examples to illustrate it, specially of singularities whose links are topological 3-manifolds. We also show through some examples how that the computations presented in these algorithms enable one to describe the link of any singularity $(S,p) \subset (\mathbf{C}^3,0)$ given by an equation $f_d(x,y,z) + f_{d+k}(x,y,z) = 0$ where f_d and f_{d+k} denote two homogeneous polynomials in $\mathbf{C}[X,Y,Z]$ with degrees d and $d+k$. As an application, we prove that the singularity with equation $(y^2 - x^2)^2 + y^4 x = 0$ gives a negative answer to a question of McEwans and Némethi ([**12**])

In section 5, we prove Lê's conjecture for the singularities $\mathcal{C}(F,k)$ obtained by taking the cyclic cover $\rho : \mathcal{C}(F,k) \to (S,p)$ of a normal surface (S,p) totally branched over the zero locus of a germ of analytic function $F : (S,p) \to (\mathbf{C},0)$ (Theorem 5.1). The link $\mathcal{L}(\mathcal{C}(F,k))$ of $\mathcal{C}(F,k)$ can be defined as the inverse image of $\mathcal{L}(S,p)$ by ρ. Let $L_F \subset \mathcal{L}(S,p)$ be the link of the curve $F^{-1}(0)$. The main argument of the proof of 5.1 is the following surprising fact (Proposition 5.3): when L_F is connected, the minimal Waldhausen decomposition of $\mathcal{L}(\mathcal{C}(F,k))$ such that the link $\rho^{-1}(L_F)$ is a Seifert fibres is also the minimal Waldhausen decomposition of $\mathcal{L}(\mathcal{C}(F,k))$.

2. Topological action of the normalization

Let (S,p) be a reduced germ of complex surface; in particular, the singularity at p is allowed to be non-isolated. One denotes by Σ_S the singular locus of S. Let us identify (S,p) with its image by an embedding $(S,p) \to (\mathbf{C}^N,0)$. The *link* $\mathcal{L}(S,p)$ of (S,p) (resp. $L(\Sigma_S,p)$ of (Σ_S,p)) is the intersection in $\mathbf{C}^N$ between S (resp. Σ_S) and a sufficiently small sphere $\mathbf{S}^{2N-1}_\varepsilon$ of radius ε centered at the origin of $\mathbf{C}^N$.

According to the cone structure theorem ([**13**]), the homeomorphism class of the pair $(\mathcal{L}(S,p), L(\Sigma_S,p))$ does not depend on N, nor on the embedding of (S,p) in $(\mathbf{C}^n,0)$, nor on ε when ε is sufficiently small.

If the singularity (S,p) is isolated, then $L(\Sigma_S,p)$ is empty. Otherwise $L(\Sigma_S,p)$ is a 1-dimensional manifold diffeomorphic to a finite disjoint union of circles. $\mathcal{L}(S,p) \smallsetminus L(\Sigma_S,p)$ is a differentiable 3-manifold and the topological singular locus of $\mathcal{L}(S,p)$ is included in $L(\Sigma_S,p)$. Note that $\mathcal{L}(S,p)$ may be a topological manifold even if the singularity (S,p) is not isolated. For example, the link of $(\{(x,y,z) \in \mathbf{C}^3 \mid x^2+y^3=0\},0)$ is homeomorphic to the sphere $\mathbf{S}^3$ whereas the singular locus is the z-axis.

In order to lighten the notations when dealing with some germ of analytic space (X,p), we often remove p from the notations when no confusion on the point p is possible, writing for example S, Σ_S, $\mathcal{L}(S)$ and $L(\Sigma_S)$ instead of (S,p), (Σ_S,p), $\mathcal{L}(S,p)$ and $L(\Sigma_S,p)$. Furthermore, we also denote by (X,p) or simply X a sufficiently small neighbourhood of p in X.

Let $(S_1,p),\ldots,(S_r,p)$ be the irreducible components of (S,p). For each $i=1,\ldots,r$, let $n_i : (\overline{S}_i,p_i) \to (S_i,p)$ be the normalisation of (S_i,p), *i.e.* the morphism, unique up to composition with an analytic isomorphism, such that n_i is proper with finite fibres, the germ $(\overline{S}_i,p_i)$ is normal, $\overline{S}_i \smallsetminus n_i^{-1}(\Sigma_{S_i})$ is dense in $\overline{S}_i$, and the restriction of n_i to $\overline{S}_i \smallsetminus n_i^{-1}(\Sigma_{S_i})$ is biholomorphic. The *normalisation* of (S,p) is the map $n : \coprod_{i=1}^r (\overline{S}_i,p_i) \to (S,p)$ defined by: $\forall\, i=1,\ldots,r$, $n_{|\overline{S}_i} = n_i$.

We call a *circle* an oriented topological space diffeomorphic to $\mathbf{S}^1 = \{z \in \mathbf{C} \mid |z|=1\}$.

Definition. — Let T be a topological space, let $C \subset T$ be a circle and let $n \geqslant 1$ be an integer. Let us choose an orientation-preserving diffeomorphic $\gamma : C \to \mathbf{S}^1$. One defines an equivalence relation $\sim$ on T by setting:

$$(x \sim y) \Longleftrightarrow \big((x=y) \text{ or } (x \in C, y \in C, \exists\, k \in \mathbf{Z} \text{ such that } \gamma(x) = e^{2ik\pi/n}\gamma(y)\,)\big)$$

One calls *n-curling on C* the projection $T \to T/{\sim}$.

Note that the homeomorphism class of the quotient space $T/{\sim}$ does not depend on the choice of γ. One denotes by $C/(n)$ the subspace $C/{\sim}$ of $T/{\sim}$.

Definition. — Let T be a topological space and let C and C' be two disjoint circles in T. Let us choose an orientation-preserving diffeomorphism $\delta : C \to C'$ and let us consider the equivalence relation $\sim'$ defined on T by:

$$(x \sim' y) \Longleftrightarrow \big((x = y) \text{ or } (x \in C,\, y \in C',\, \delta(x) = y\big)$$

One calls *identification of the two circles C and C'* the projection $T \to T/\sim'$.

Note that the homeomorphism class of $T/\sim'$ does not depend on the choice of δ. When s is an integer $\geqslant 3$, the identification of s circles in T is defined from this by induction.

Let (S, p) be a singularity of complex surface and let $n : \coprod_{i=1}^{r}(\overline{S}_i, p_i) \to (S, p)$ be its normalisation. According to the theory of semialgebraic or subanalytic neighbourhoods (see [**3**] and [**7**]), there exists a subanalytic rug function $\phi : S \to \mathbf{R}$ for $\{p\}$ in S such that for $\varepsilon > 0$ sufficiently small, $\mathcal{L}(S, p) = \phi^{-1}(\varepsilon)$. As n is analytic, $\phi \circ n$ is a subanalytic rug function for $\coprod_{i=1}^{r}\{p_i\}$ in $\coprod_{i=1}^{r}(S_i, p_i)$. Therefore, if $\varepsilon > 0$ is sufficiently small, then $(\phi \circ n)^{-1}(\varepsilon)$ can be taken as the link of $\coprod_{i=1}^{r}(S_i, p_i)$. In particular, we have that $n^{-1}(\mathcal{L}(S, p)) = \coprod_{i=1}^{r} \mathcal{L}(S_i, p_i)$

Proposition 2.1

(1) *n is an homeomorphism over the complementary of a tubular neighbouhood N of $L(\Sigma_S)$ in $\mathcal{L}(S, p)$.*

(2) *Let $\Sigma_S = \cup_{k=1}^{s}\Gamma_k$, with Γ_k irreducible, and for each k, let $n^{-1}(\Gamma_k) = \cup_{j=1}^{l_k}\Delta_j^k$ with Δ_j^k irreducible. Let a_j^k be the degree of n on Δ_j^k. Then the restriction of n to N is the composition of the a_j^k-curlings on the circles $L(\Delta_j^k)$ for $k = 1, \ldots, s$ and $j = 1, \ldots, l_k$ and of the identifications of the l_k circles $L(\Delta_j^k)/(a_j^k)$ for $k = 1, \ldots, s$.*

Proof. — This follows from the fact that, topologically, the normalisation just separates the branches of the surface at each of its points. □

Remark. — $\mathcal{L}(S, p)$ is a topological manifold if and only if for each irreducible component Γ_k of Σ_S, $l_k = 1$ and $a_1^k = 1$.

Let (S, p) be a normal singularity of complex surface, and let $\pi : Z \to S$ be a resolution of (S, p) whose exceptional divisor $\pi^{-1}(p)$ has normal crossings. The dual graph G_π of the exceptional divisor $\pi^{-1}(p)$ with vertices weighted by the self-intersections and the genus of the irreducible components of $\pi^{-1}(p)$ completely determines the homeomorphism class of $\mathcal{L}(S, p)$; namely, $\mathcal{L}(S, p)$ is homeomorphic to the boundary of the 4-dimensional manifold obtained from G_π by a plumbing process, as described in [**15**].

Let $C \subset S$ be a germ of curve on $(\overline{S}, p)$. One calls *embedded resolution* of C any resolution $\pi : Z \to S$ of (S, p) such that the total transform of C by π has normal crossings. Such a π is obtained by composing any resolution of (S, p) with a suitable finite sequence of blowing-up of points. A *resolution graph* of C is a resolution graph G_π of such a π to which one adds a stalk (see figure 1) for each component of the strict transform of C by π at the corresponding vertex. (usually one uses arrows instead of stalks, but arrows will be used later to represent the components of a multilink associated with a germ of function).

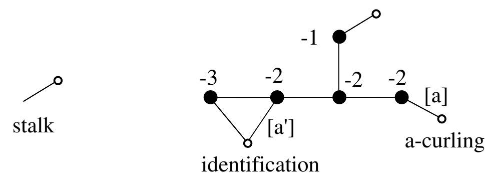

FIGURE 1

Let now (S,p) be an arbitrary singularity of complex surface and let $n : \coprod_{i=1}^{r}(\overline{S}_i, p_i) \to (S,p)$ be its normalization. For each $i \in \{1,\dots,r\}$, let us choose an embedded resolution π_i of the germ of curve $(n^{-1}(\Sigma_S), p_i) \subset (\overline{S}_i, p_i)$ and let G_{π_i} be the corresponding resolution graph of $(n^{-1}(\Sigma_S), p_i)$. Then, according to Proposition 2.1, the homeomorphism class of the link $\mathcal{L}(S,p)$ is encoded in the *generalised plumbing graph* of $\mathcal{L}(S,p)$ obtained from the disjoint union of the graphs G_{π_i} by performing the following operation for each irreducible component Γ_k of Σ_S. Using again the notations of Proposition 2.1, if $a_j^k \neq 1$, the stalk corresponding to Δ_j^k is weighted by $[a_j^k]$ in order to symbolize the quotient circle $L(\Delta_j^k)/(a_j^k)$. Then the extremities of these l_k stalks are joined in a single extremity which symbolizes the identification of the l_k circles links $L(\Delta_j^k)$. if $l_k = 1$ and $a_1 = 1$, one simply remove from $\coprod_{i=1}^{r} G_i$ the stalk representing $L(\Delta_1^i)$.

Example. — Let $(S,0)$ be the germ of hypersurface at the origin of $\mathbf{C}^3$ with equation $f(x,y) + zg(x,y) = 0$, where $f : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ and $g : (\mathbf{C}^2,0) \to (\mathbf{C},0)$ are two analytic germs which have no irreducible components in common. Let $f : \mathcal{U} \to \mathbf{C}$ and $g : \mathcal{U} \to \mathbf{C}$ be some representatives of the germs f and g. We will describe a generalized resolution graph of the link $\mathcal{L}(S,0)$ from a a resolution of the meromorphic function $h = (f:g) : \mathcal{U} \to \mathbf{P}^1$, *i.e.* a finite sequence $\rho : \widehat{\mathcal{U}} \to \mathcal{U}$ of blowing-up of point such that the map $\widehat{h} : \widehat{\mathcal{U}} \to \mathbf{P}^1$ given by $\widehat{h} = h \circ \rho$ is well defined (see for instance [**9**]).

Let Z_0 (resp. Z_∞) be the union of the irreducible components of the exceptional divisor $\rho^{-1}(0)$ such that $\widehat{h}(Z_0) = (0:1)$ (resp. $\widehat{h}(Z_\infty) = (1:0)$). A component E of $\rho^{-1}(0)$ is *dicritical* if the restriction of $\widehat{h}$ to E is not constant. One denotes by D the union of the dicritical components.

If necessary, one composes ρ with a finite sequence of blowings-up in such a way that the new morphism, again denoted by ρ, verifies that the strict transform of $f^{-1}(0)$ by ρ does not intersect D.

Let $Z_1, \dots, Z_m$ be the connected components of Z_0. For each $i = 1,\dots,m$, one denotes by $\widehat{U}_i$ a small regular neighbourhood of Z_i in $\widehat{U}$. As the intersection form restricted to Z_i is negative definite, one obtains a germ of normal surface $(\overline{S}_i, p_i)$ by contracting Z_i to a point p_i ([**5**]). Then the projection $c_i : \widehat{\mathcal{U}}_i \to \overline{S}_i$ is a resolution of $(\overline{S}_i, p_i)$.