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RESIDUES OF CHERN CLASSES ON SINGULAR VARIETIES

by

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Abstract. — For a collection of sections of a holomorphic vector bundle over a complete intersection variety, we give three expressions for its residues at an isolated singular point. They consist of an analytic expression in terms of a Grothendieck residue on the variety, an algebraic one as the dimension of a certain complex vector space and a topological one as a mapping degree. Some examples are also given.

***Résumé* (Résidus de classes de Chern sur les variétés singulières). —** Étant donnée une famille de sections d'un fibré vectoriel complexe sur une variété intersection complète, on donne trois expressions pour le résidu en un point singulier isolé. Elles consistent en une expression analytique en termes d'un résidu de Grothendieck sur la variété, une expression algébrique comme dimension d'un certain espace vectoriel complexe et une expression topologique comme degré d'une application. Quelques exemples sont aussi donnés.

This is a partially expository article, in which we give various expressions for the residues of Chern classes of vector bundles, mainly over complete intersection varieties.

Let E be a complex vector bundle of rank r over some reasonable space X of real dimension m. For an ℓ-tuple of sections $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of E, we denote by $S(\boldsymbol{s})$ its singular set, *i.e.*, the set of points where the s_i's fail to be linealy independent. Let $c^i(E)$ denote the i-the Chern class of E, which is in $H^{2i}(X)$. For $i \geqslant r - \ell + 1$, there is a natural lifting $c^i_S(E, \boldsymbol{s})$ in $H^{2i}(X, X \smallsetminus S)$ of $c^i(E)$, $S = S(\boldsymbol{s})$. We call $c^i_S(E, \boldsymbol{s})$ the localization of $c^i(E)$ at S with respect to $\boldsymbol{s}$. Suppose S is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then, by the Alexander homomorphism $H^{2i}(X, X \smallsetminus S) \to H_{m-2i}(S) = \oplus_\lambda H_{m-2i}(S_\lambda)$, the class $c^i_S(E, \boldsymbol{s})$ determines, for

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each λ, the "residue" $\mathrm{Res}_{c^i}(\boldsymbol{s}, E; S_\lambda)$ in $H_{m-2i}(S_\lambda)$. If X is compact, we have the "residue formula"

$$\sum_\lambda (\iota_\lambda)_* \mathrm{Res}_{c^i}(\boldsymbol{s}, E; S_\lambda) = c^i(E) \frown [X],$$

where $\iota_\lambda \hookrightarrow X$ denotes the inclusion and $[X]$ the fundamental class of X. The formula itself is of rather trivial nature. However, everytime we have an explicit expression for the residues, it becomes really an interesting one.

In this article, we consider the case where X is a complex manifold M or a (locally) complete intersection variety V of dimension n. We also assume that $r-\ell+1 = n$ and look at $c^n(E)$ so that the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$ under consideration is a number. In tha case S_λ consists of an isolated point p, we give analytic, algebraic and topological expressions for $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; p)$. As a consequence we have the fact that these three expressions are the same, which is rather well-known in some cases, in particular in the case $X = M$, $r = n$ and $\ell = 1$ (see, *e.g.*, [**DA**], [**GH**], [**O**]). For the analytic expression, we quote results of [**Su4**] and for the algebraic one we try to give a complete proof. The proof for the topological one is not so difficult and we only state the outline.

In Section 1, we recall the residues and describe them in the case we consider. This is done in the framework of Chern-Weil theory adapted to the Čech-de Rham cohomology. In Section 2, we give fundamental properties of residues at isolated singularities. In particular, we show that they are positive integers and satisfy the "conservation law" under perturbations of sections. In Section 3, we give an analytic expression of the residue as a Grothendieck residue (on a variety), quoting the results in [**Su4**]. After we recall some commutative algebra in Section 4, we give an algebraic expression of the residue as the dimension of some complex vector space in Section 5. The proof is done by showing that this algebraic invariant also satisfies the conservation law. It should be noted that the idea of proof is inspired by [**EG1**] and [**Lo**, Ch. 4]. In Section 6, we give a topological expression as the degree of some map of the link of the singularity to the Stiefel manifold. This is also done by noting that the degree satisfies the conservation law. Finally in Section 7, we give some examples and applications.

After the preparation of the manuscript, the author's attention was drawn to a recent preprint of W. Ebeling and S.M. Gusein-Zade [**EG2**]. They consider also characteristic numbers (not only Chern classes) and define the index of a collection of sections topologically. Their algebraic formula in Theorem 2 is more general than the one in Theorem 5.5 below. They also give a formula (Theorem 4), which corresponds to the one in Theorem 5.8 below, for collections of 1-forms.

1. Residues of Chern classes

We refer to [**Su2**, Ch. IV, 2, Ch. VI, 4] and [**Su4**] for details of the material in this section.

1a. Non-singular base spaces. — Let M be a complex manifold of dimension n and E a (C^∞, for the moment) complex vector bundle of rank r over M. Then, for $i = 1, \dots, r$, we have the i-th Chern class $c^i(E)$ in $H^{2i}(M)$. If we use the obstruction theory, it is the primary obstruction to constructing $r - i + 1$ sections linearly independent everywhere (see, *e.g.*, [**St**]). The Chern-Weil theory provides us with a canonical way of constructiong a closed $2i$-form representing the class $c^i(E)$ in the de Rham cohomology. To be a little more precise, let ∇ be a connection for E. For the i-th Chern polynomial c^i, we have a closed $2i$-form $c^i(\nabla)$ on M. Moreover, for two connections ∇ and ∇', we have the "Bott difference form" $c^i(\nabla, \nabla')$, which is a $(2i-1)$-form satisfying

$$c^i(\nabla', \nabla) = -c^i(\nabla, \nabla') \quad \text{and} \quad d\, c^i(\nabla, \nabla') = c^i(\nabla') - c^i(\nabla).$$

Then the class of $c^i(\nabla)$ is independent of the choice of ∇ and is equal to $c^i(E)$. Hereafter we assume that $r \geqslant n$ and look at the class $c^n(E)$, which is in the cohomology of M of the top dimension.

For an ℓ-tuple of sections $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of E, we denote by $S(\boldsymbol{s})$ its singular set, *i.e.*, the set of points where $s_1, \dots, s_\ell$ fail to be linearly independent. Suppose we have such an $\boldsymbol{s}$ with $\ell = r - n + 1$ and set $S = S(\boldsymbol{s})$. Then there is the "localization" $c^n_S(E, \boldsymbol{s})$ in $H^{2n}(M, M \smallsetminus S; \mathbb{C})$, with respect to $\boldsymbol{s}$, of the n-th Chern class $c^n(E)$, which is described as follows.

Letting $U_0 = M \smallsetminus S$ and U_1 a neighborhood of S, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. Recall that, in the Čech-de Rham cohomology for the covering $\mathcal{U}$, the class $c^n(E)$ is represented by a cocycle of the form

$$c^n(\nabla_\star) = (c^n(\nabla_0), c^n(\nabla_1), c^n(\nabla_0, \nabla_1)), \tag{1.1}$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1, respectively. If we take as ∇_0 an $\boldsymbol{s}$-trivial connection (*i.e.*, a connection ∇_0 with $\nabla_0(s_i) = 0$ for $i = 1, \dots, \ell$), then $c^n(\nabla_0) = 0$ and the cocycle naturally defines a class in the relative cohomology $H^{2n}(M, M \smallsetminus S; \mathbb{C})$, which we denote by $c^n_S(E, \boldsymbol{s})$. It is sent to $c^n(E)$ by the canonical homomorphism $j^* : H^{2n}(M, M \smallsetminus S; \mathbb{C}) \to H^{2n}(M, \mathbb{C})$.

Suppose now that $S = S(\boldsymbol{s})$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then for each λ, the class $c^n_S(E, \boldsymbol{s})$ defines a number, which we call the residue of $\boldsymbol{s}$ at S_λ with respect to c^n and denote by $\operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$. It is also briefly called a residue of $c^n(E)$. For each λ, we choose a neighborhood U_λ of S_λ in U_1 so that the U_λ's are mutually disjoint, and let R_λ be a real $2n$-dimensional manifold with C^∞ boundary ∂R_λ in U_λ containing S_λ in its interior. Then the residue is given by

$$\operatorname{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda) = \int_{R_\lambda} c^n(\nabla_1) - \int_{\partial R_\lambda} c^n(\nabla_0, \nabla_1). \tag{1.2}$$

We have the "residue formula" (*cf.* [**Su2**, Ch. III, Theorem 3.5]):

Proposition 1.3. — *If R is a compact real $2n$-dimensional manifold with C^∞ boundary containing S in its interior, then*

$$\sum_\lambda \mathrm{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda) = \int_R c^n_R(E, \boldsymbol{s}),$$

where the right hand side is defined as that of (1.2) with ∇_0 an $\boldsymbol{s}$-trivial connection for E on a neighborhood of ∂R, ∇_1 a connection for E on a neighborhood of R and R_λ replaced by R.

In particular, if M is compact, the right hand side is equal to $\int_M c^n(E)$.

Remark 1.4. — Comparing with the obstruction theoretic definition of Chern classes, we see that the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E; S_\lambda)$ is in fact an integer. However, in the sequel we prove this fact more directly in the pertinent cases.

1b. Singular base spaces. — Let V be an analytic variety of pure dimension n in a complex manifold W of dimension $n+k$. We denote by $\mathrm{Sing}(V)$ the singular set of V and let $V' = V \smallsetminus \mathrm{Sing}(V)$ be the non-singular part.

Let S be a compact set in V (V may not be compact). We assume that S has a finite number of connected components, $S \supset \mathrm{Sing}(V)$ and that S admits a regular neighborhood in W. Let $\widetilde{U}_1$ be a regular neighborhood of S in W and $\widetilde{U}_0$ a tubular neighborhood of $U_0 = V \smallsetminus S$ in W. We consider the covering $\mathcal{U} = \{\widetilde{U}_0, \widetilde{U}_1\}$ of the union $\widetilde{U} = \widetilde{U}_0 \cup \widetilde{U}_1$, which may be assumed to have the same homotopy type as V.

For a complex vector bundle E over $\widetilde{U}$ of rank r ($\geqslant n$), the n-th Chern class $c^n(E)$ is in $H^{2n}(\widetilde{U}) \simeq H^{2n}(V)$. The corresponding class in $H^{2n}(V)$ is denoted by $c^n(E|_V)$. The class $c^n(E)$ is represented by a Čech-de Rham cocycle $c^n(\nabla_\star)$ on $\mathcal{U}$ given as (1.1) with ∇_0 and ∇_1 connections for E on $\widetilde{U}_0$ and $\widetilde{U}_1$, respectively. Note that it is sufficient if ∇_0 is defined only on U_0, since there is a C^∞ retraction of $\widetilde{U}_0$ onto U_0. Suppose we have an ℓ-tuple $\boldsymbol{s} = (s_1, \dots, s_\ell)$ of C^∞ sections linearly independent everywhere on U_0, $\ell = r - n + 1$, and let ∇_0 be $\boldsymbol{s}$-trivial. Then we have the vanishing $c^n(\nabla_0) = 0$ and the above cocycle $c^n(\nabla_\star)$ defines a class $c^n_S(E|_V, \boldsymbol{s})$ in $H^{2n}(V, V \smallsetminus S; \mathbb{C})$. It is sent to $c^n(E|_V)$ by the canonical homomorphism $j^* : H^{2n}(V, V \smallsetminus S; \mathbb{C}) \to H^{2n}(V, \mathbb{C})$.

Let $(S_\lambda)_\lambda$ be the connected components of S. Then, for each λ, $c^n_S(E|_V, \boldsymbol{s})$ defines the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$. For each λ, we choose a neighborhood $\widetilde{U}_\lambda$ of S_λ in $\widetilde{U}_1$, so that the $\widetilde{U}_\lambda$'s are mutually disjoint. Let $\widetilde{R}_\lambda$ be a real $2(n+k)$-dimensional manifold with C^∞ boundary $\partial\widetilde{R}_\lambda$ in $\widetilde{U}_\lambda$ containing S_λ in its interior such that $\partial\widetilde{R}_\lambda$ is transverse to V. We set $R_\lambda = \widetilde{R}_\lambda \cap V$. Then the residue is a number given by a formula as (1.2). We also have the residue formula:

Proposition 1.5. — *If $\widetilde{R}$ is a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\widetilde{U}$ containing S in its interior such that $\partial\widetilde{R}$ is transverse to V,*

$$\sum_\lambda \mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda) = \int_R c^n_R(E|_V, \boldsymbol{s}),$$

where the right hand side is defined as that of (1.2) with ∇_0 an $\boldsymbol{s}$-trivial connection for E on a neighborhood of ∂R in V, ∇_1 a connection for E on a neighborhood of $\widetilde{R}$ in W and R_λ replaced by R, $R = \widetilde{R} \cap V$.

In particular, if V is compact, the right hand side is equal to $\int_V c^n(E)$.

Remarks

(1) If S_λ is in the non-singular part V', $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$ coincides with the one defined in (1a) and if V itself is non-singular, Proposition 1.5 reduces to Proposition 1.3.

(2) If $\boldsymbol{s}$ extends to an ℓ-tuple $\widetilde{\boldsymbol{s}}$ of sections of E linearly independent everywhere on $\widetilde{U}_\lambda$, we may let both ∇_0 and ∇_1 equal to an $\widetilde{\boldsymbol{s}}$-trivial connection so that we have $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda) = 0$.

(3) As in the case of non-singular base spaces (*cf.* Remark 1.4), the residue $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; S_\lambda)$ is in fact an integer. In the sequel we prove this fact more directly in the pertinent cases.

1c. Residues at an isolated singularity. — Let V be a subvariety of dimension n in a complex manifold W of dimension $n+k$, as before. We do not exclude the case $k=0$, where $V = W$ is a complex manifold of dimension n.

Suppose now that V has at most an isolated singularity at p and let E be a holomorphic vector bundle of rank r ($\geqslant n$) on a small coordinate neighborhood $\widetilde{U}$ of p in W. Sometimes we identify $\widetilde{U}$ with a neighborhood of 0 in $\mathbb{C}^{n+k}$ and p with 0. We may assume that E is trivial and let $\boldsymbol{e} = (e_1, \dots, e_n)$ be a holomorphic frame of E on $\widetilde{U}$. Let $\ell = r - n + 1$ and suppose we have an ℓ-tuple of holomorphic sections $\widetilde{\boldsymbol{s}}$ of E on $\widetilde{U}$. Suppose that $S(\widetilde{\boldsymbol{s}}) \cap V = \{p\}$. Then we have $\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p)$ with $\boldsymbol{s} = \widetilde{\boldsymbol{s}}|_V$. Let $\widetilde{R}$ be a compact real $2(n+k)$-dimensional manifold with C^∞ boundary in $\widetilde{U}$ containing p in its interior such that $\partial\widetilde{R}$ is transverse to V and set $R = \widetilde{R} \cap V$. We also set $U = \widetilde{U} \cap V$ and let ∇_0 be an $\boldsymbol{s}$-trivial connection for E on $U \smallsetminus \{p\}$. We choose ∇_1 to be $\boldsymbol{e}$-trivial. Then we have $c^n(\nabla_1) = 0$ and

$$\mathrm{Res}_{c^n}(\boldsymbol{s}, E|_V; p) = -\int_{\partial R} c^n(\nabla_0, \nabla_1). \tag{1.6}$$

In the subsequent sections, we give various expressions of this number.

2. Fundamental properties of the residues

2a. Non-singular base spaces. — In the situation of (1c), suppose $V = W = M$ is a complex manifold of dimension n and write $\widetilde{U}$ and $\widetilde{\boldsymbol{s}}$ by U and $\boldsymbol{s}$, respectively. Thus our assumption is $S(\boldsymbol{s}) = \{p\}$.

Let us first assume that $r = n$. Thus $\ell = 1$ and we have only one section s. We write $s = \sum_{i=1}^n f_i\, e_i$ with f_i holomorphic functions on U.