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2-DIMENSIONAL VERSAL S_4-COVERS AND RATIONAL ELLIPTIC SURFACES

by

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Abstract. — We introduce the notion of a versal Galois cover, and study versal S_4-covers explicitly. Our goal of this article is to show that two S_4-covers arising from certain rational elliptic surfaces are versal.

***Résumé* (S_4-revêtements galoisiens versels de dimension 2 et surfaces rationnelles elliptiques)**
On introduit la notion de revêtement galoisien versel et on étudie explicitement les S_4-revêtements galoisiens. Le but de cet article est de montrer que deux S_4-revêtements galoisiens obtenus à partir de certaines surfaces elliptiques rationnelles sont versels.

Introduction

Let G be a finite group. Let X and Y be normal projective varieties. X is called a G-cover of Y if there exists a finite surjective morphism $\pi : X \to Y$ such that the induced inclusion morphism $\pi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ gives a Galois extension with $\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$, where $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ denote the rational function fields of X and Y, respectively.

G-covers have been used in various branches of algebraic geometry and topology, *e.g.*, to construct algebraic varieties having the prescribed invariants, to study the topology of the complement to a reduced plane algebraic curve, and so on. In this article, our main concern is not applications of G-covers, but G-covers themselves.

One of fundamental problems in the study of G-covers is to give an explicit "bottom-to-top" method in constructing G-covers from some geometric data of the base variety Y or intermediate covers, *i.e.*, covers corresponding to the intermediate field between $\mathbb{C}(X)$ and $\mathbb{C}(Y)$. This point of view resembles the constructive aspects of

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the inverse Galois problem: to construct a field extension of $\mathbb{Q}$ having a prescribed group as its Galois group over $\mathbb{Q}$.

In the investigation of the inverse Galois problem, many works have been done about "generic polynomials or versal polynomials" for these twenty years (see [**3**] for detail references, for example). The main purpose of this article is to make an analogous *geometric study* of them. Let us begin with the definition of a versal G-cover.

Definition 0.1. — A G-cover $\varpi : X \to M$ is said to be versal if it satisfies the following property:

For any G-cover $\pi : Y \to Z$, there exist a rational map $\nu : Z \cdots \to M$ and a Zariski open set U in Z such that

(i) $\nu|_U : U \to M$ is a morphism, and
(ii) $\pi^{-1}(U)$ is birational to $U \times_M X$ over U.

Note that we do not assume any uniqueness for ϖ and ν. Also we do not assume that ν is dominant. One could say that a versal G-cover is a geometric realization of the Galois closure of a versal G-polynomial introduced in [**1**].

Intuitively, any G-cover is obtained as *rational* pull-back of ϖ, if a versal G-cover exists. It is known that a versal G-cover exists for any G (see [**9**], [**10**]). Concretely, let $n = \#(G)$ and let $X = (\mathbb{P}^1)^n$ be the n-ple direct product of $\mathbb{P}^1$. By using the regular representation of G, one can regard G as a transitive subgroup of S_n (the symmetric group of n letters), and obtain a natural G-action on X by the permutation of the coordinates. Let $M := X/G$ be the quotient variety with respect to this action, and we denote the quotient morphism by $\varpi : X \to M$. Then we have

***Theorem 0.1* (Namba [9], [10])**. — *$\varpi : X \to M$ is a versal G-cover.*

By Theorem 0.1, the existence of a versal G-cover is assured for any G. Namba's model, however, has too large dimension to use it to consider concrete problems. Also his construction is "top-to-bottom," *i.e.*, the one to find a variety with a natural G-action first, and then to take the quotient with respect to this action. This approach is different from our viewpoint. This leads us to pose the following question:

Question 0.1. — Find a tractable versal G-cover (via a "bottom-to-top" construction if possible).

In order to obtain a tractable versal G-cover, it is natural to consider such cover of as small dimension as possible. To formulate our problem along this line, the notion of the essential dimension of G introduced by Buhler and Reichstein in [**1**] is at our disposal. The essential dimension of G gives the lower bound of dimensions of versal G-covers and it is denoted by $\mathrm{ed}_k(G)$, where k is the base field of variety ($k = \mathbb{C}$ in our case). We refer to [**1**] about details on $\mathrm{ed}_k(G)$, and put here some of the properties and results about $\mathrm{ed}_{\mathbb{C}}(G)$:

– $\mathrm{ed}_{\mathbb{C}}(G) = 1$ if and only if G is either a cyclic group $\mathbb{Z}/n\mathbb{Z}$ or a dihedral group D_{2r} (r: odd) of order $2r$. Versal G-covers of dimension 1 are classically well-known (see § 2 or [**1**]).

– $\mathrm{ed}_{\mathbb{C}}(G) = 2$ for $G = S_4, A_4, A_5, S_5$, where S_n and A_n denote the symmetric and alternating groups of n letters, respectively.

– $\mathrm{ed}_{\mathbb{C}}(G)$ is equal to the smallest dimension of a versal G-cover (Theorem 7.5 in [**1**]).

The purpose of this article is to study versal S_4-covers of dimension 2 as a first step of the study of versal G-covers. In § 1, we summarize for a method to deal with S_4-covers developed in [**15**]. In § 2, we give two examples of S_4-covers using this method. We denote them by $\pi_{431} : S_{431} \to \Sigma_{431}$ and $\pi_{9111} : S_{9111} \to \Sigma_{9111}$. Both of them are constructed from certain rational elliptic surfaces in a canonical way. Both of the actions of the Galois groups S_4 on S_{431} and S_{9111} are described by the language of the Mordell-Weil groups of the corresponding elliptic surfaces by the same idea. Our goal of this article is to prove the following:

Theorem 0.2. — *Both* $\pi_{431} : S_{431} \to \Sigma_{431}$ *and* $\pi_{9111} : S_{9111} \to \Sigma_{9111}$ *are versal* S_4*-covers.*

The rest of this article is devoted to proving this theorem. We first show that π_{9111} is versal by using Tsuchihashi's result in [**17**] in § 3. In § 4, we explain a method for a top-to-bottom method in constructing of a versal G-cover by using a linear representation of G. The method seems to be well-known to the specialists who are working on generic polynomials or versal polynomials. In fact, it is essentially used in [**1**]. Yet we put it here since we need it to prove the versality for π_{431}. We give several examples in § 5 by using this method. In § 6, we prove the versality for π_{431} by comparing S_{431} with an example in § 5.

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1. S_4-covers

In [**15**], the author has developed a method in studying Galois covers having S_4 as their Galois groups. We here explain it briefly (see [**15**] for a proof). For a finite surjective morphism $\pi : X \to Y$, the branch locus of π is the subset of Y given by

$$\{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}.$$

We denote it by $\Delta(X/Y)$ or Δ_π.

Let $\pi : X \to Y$ be an S_4-cover. Let $V_4 (\cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2})$ be the subgroup given by

$$\{1, (12)(34), (13)(24), (14)(23)\},$$

and let $\mathbb{C}(X)^{V_4}$ be the V_4-invariant subfield of $\mathbb{C}(X)$. We denote the $\mathbb{C}(X)^{V_4}$-normalization of Y by $D(X/Y, V_4)$. There are canonical morphisms:

$$\beta_1(\pi, V_4) : D(X/Y, V_4) \longrightarrow Y, \quad \beta_2(\pi, V_4) : X \longrightarrow D(X/Y, V_4).$$

Note that $\beta_2(\pi, V_4)$ is a $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$-cover, while $\beta_1(\pi, V_4)$ is an S_3-cover, where S_3 denotes the symmetric group of 3 letters.

Proposition 1.1. — *Let $f : Z \to Y$ be an S_3-cover of Y. Suppose that Z is smooth and there exist three different reduced divisors, D_1, D_2 and D_3 on Z satisfying the following conditions:*

(i) *There is no common component among D_1, D_2 and D_3. Put* $\mathrm{Gal}(Z/Y) = S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle$, *then $(i-a)$ $D_1^{\sigma} = D_2$ and $D_3^{\sigma} = D_3$, and $(i-b)$ $D_1^{\tau} = D_2$, $D_2^{\tau} = D_3$, $D_3^{\tau} = D_1$. (D^{σ} and D^{τ} denote the pull-back of D by σ and τ, respectively).*

(ii) *There exists a line bundle, $\mathbb{L}$, such that D_1 is linearly equivalent to $2\mathbb{L}$.*

Then there exists an S_4-cover $\pi : X \to Y$ satisfying (i) $D(X/Y, V_4) = Z$ and (ii) $\Delta(X/Z) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

2. S_4-covers arising from certain rational elliptic surfaces

In this section, we make use of various results in the theory of elliptic surfaces freely in order to construct two example which play main roles in this article. See for [**4**], [**6**], [**7**] and [**13**] for the details about the theory of elliptic surfaces. Note that our method in this section can be generalized to any elliptic surface $\varphi : S \to \mathbb{P}^1$ with 3-torsion

2.1. The surface S_{431}. — Let $\varphi : X_{431} \to \mathbb{P}^1$ be a rational elliptic surface obtained by blowing up base points $q : X_{431} \to \mathbb{P}^2$ of the pencil of cubic curves

$$\Lambda : \{\lambda_0(X_0X_1X_2) + \lambda_1(X_0 + X_1 + X_2)^3 = 0\}_{[\lambda_0,\lambda_1]\in\mathbb{P}^1},$$

where X_0, X_1, X_2 are homogeneous coordinates of $\mathbb{P}^2$. The notation X_{431} is due to [**7**]. It is known that $\varphi : X_{431} \to \mathbb{P}^1$ satisfies the following properties(see [**7**]):

– The Mordell-Weil group, $\mathrm{MW}(X_{431})$, is isomorphic to $\mathbb{Z}/3\mathbb{Z}$; and we denote its elements by O, s_1 and s_2.

– φ has three singular fibers and their types are of I_1, I_3 and IV^*.

We may assume that the three singular fibers , s_1 and s_2 sit in X_{431} as in Figure 1 below. The curves O, s_1, s_2, $C_{2,i}$ $(i = 0, 1, 2, 4, 5, 6)$ are the exceptional curves of q.

Let

– σ_{φ} = the inversion morphism with respect to the group law

– τ_{s_i} = the translation by s_i.

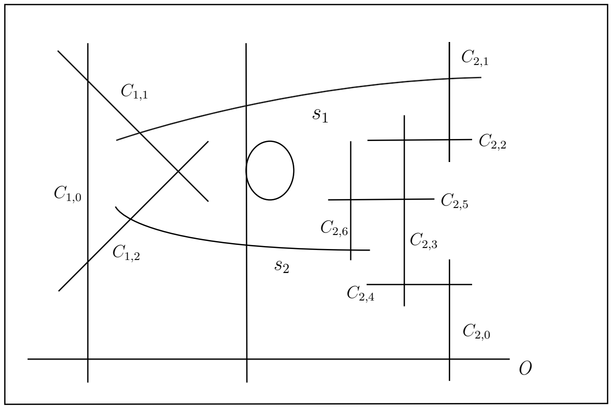

FIGURE 1

Both σ_φ and τ_{s_1} are fiber preserving automorphisms on X_{431} such that $\sigma_\varphi^2 = \tau_{s_1}^3 = (\sigma_\varphi\tau_{s_1})^2 = 1$. Hence σ_φ and τ_{s_1} define an S_3-action on X_{431}. We put $\Sigma_{431} = X_{431}/S_3$, and denote its quotient morphism by $f_{431} : X_{431} \to \Sigma_{431}$. On a smooth fiber of φ, this S_3-action is a natural one: the S_3-action induced by the inversion and the translation by a 3-torsion on an elliptic curve.

Lemma 2.1. — *The S_3-action on the singular fibers are described as follows:*

I_1-fiber: σ_φ and τ_{s_1} give non-trivial automorphisms. By taking a suitable local coordinate (z_1, z_2) around the node P, they are described as follows:

$$\begin{aligned} \sigma_\varphi &: (z_1, z_2) \longmapsto (z_2, z_1), \\ \tau_{s_1} &: (z_1, z_2) \longmapsto (\omega z_1, \omega^2 z_2), \end{aligned}$$

where $P := (0,0)$ and $\omega = \exp(2\pi\sqrt{-1}/3)$.

I_3-fiber: No irreducible component is pointwise fixed. σ_φ^* and $\tau_{s_1}^*$ permute the irreducible components as follows:

$$\sigma_\varphi^* : \begin{matrix} C_{1,0} \mapsto C_{1,0}, \\ C_{1,1} \mapsto C_{1,2}, \\ C_{1,2} \mapsto C_{1,1}, \end{matrix} \quad \tau_{s_1}^* : \begin{matrix} C_{1,0} \mapsto C_{1,2}, \\ C_{1,1} \mapsto C_{1,0}, \\ C_{1,2} \mapsto C_{1,1}. \end{matrix}$$

IV^*-fiber: $C_{2,4}$ is the unique component which is pointwise fixed by σ_φ and no irreducible component is pointwise fixed by τ_{s_1}. σ_φ^* and $\tau_{s_1}^*$ permute irreducible components as follows:

$$\sigma_\varphi^* : \begin{matrix} C_{2,1} \mapsto C_{2,6},\ C_{2,2} \mapsto C_{2,5},\ C_{2,3} \mapsto C_{2,3}, \\ C_{2,4} \mapsto C_{2,4},\ C_{2,0} \mapsto C_{2,0}, \end{matrix}$$