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ALGORITHMS AND MODULI SPACES FOR DIFFERENTIAL EQUATIONS

by

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Abstract. — This article discusses second and third order differential operators. We will define standard operators, and prove that every differential operator with finite differential Galois group is a so-called pullback of some standard operator. We will also give an algorithm concerning certain field extensions, associated with algebraic solutions of a Riccati equation.

***Résumé* (Algorithmes et espaces modulaires pour les équations différentielles)**
Cet article s'intéresse aux opérateurs différentiels de deuxième et troisième ordre. Nous introduisons une notion d'opérateur standard, et montrons que tout opérateur différentiel de groupe de Galois différentiel fini est image inverse d'un opérateur standard. Nous donnons aussi un algorithme concernant certaines extensions de corps, associées à des solutions algébriques d'une équation de Riccati.

1. Field extensions for Riccati solutions

In this section we consider second order linear differential equations of the form $L : y'' = ry,\ r \in k(x)$. Here $k(x)$ is a *differential field* of characteristic zero, with derivation $\frac{d}{dx}$. The field of constants k is not supposed to be algebraically closed. We will denote its algebraic closure by $\bar{k}$. The differential Galois theory gives us an extension $\bar{k}(x) \subset K$, with K the so called *Picard-Vessiot extension*, which is the minimal differential field extension of $\overline{k}(x)$ which contains a basis $\{y_1, y_2\}$ (over $\bar{k}$) of solutions of L. The solution space $\bar{k}\langle y_1, y_2\rangle := \overline{k}y_1 + \overline{k}y_2 \subset K$ will be denoted V.

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The automorphisms of $K/\bar{k}(x)$ which commute with the differentiation constitute the *differential Galois group* G.

An interesting class of solutions are the so called *Liouvillian solutions*. These are solutions which lie in a Liouvillian extension of $\bar{k}(x)$, which roughly means they can be written down quite explicitly. For a precise definition of a (generalized) Liouvillian extension, see [**Kap76**, p. 39]. Related to this is the *Riccati equation*, denoted R_L, which is an equation depending on L with as solutions elements of the form $u = \frac{y'}{y}$, with y a solution of L. In our case it is the equation $u^2 + u' = r$. We have the following facts (see [**vdPS03**, p. 35,104]).

Fact 1.1. — *$u \in K$ is a solution of $R_L \iff u = \frac{y'}{y}$, for some $y \in V$.*

Fact 1.2. — *$u = \frac{y'}{y}$ is a solution of R_L, algebraic of degree m over $\bar{k}(x) \iff$ The stabilisor in G of the line $\bar{k} \cdot y$ is a subgroup of index m.*

The next fact is concerned with Liouvillian solutions of L.

Fact 1.3. — *L has a Liouvillian solution $\iff R_L$ has an algebraic solution.*

Let u be an algebraic solution of R_L of minimal degree over $\overline{k}(x)$. We define the field k' to be the minimal field in $\overline{k}$ such that the coefficients of the minimal polynomial of u over $\bar{k}(x)$ are elements of $k'(x)$. We want to determine k' as explicit as possible. In [**HvdP95**] bounds on the degree $[k' : k]$ are given, depending on the differential Galois group G of L. We consider G as a subgroup of $\mathrm{GL}_2(\bar{k})$ by its action on y_1, y_2. It is known that G is an algebraic subgroup of $\mathrm{GL}_2(\bar{k})$. Note that changing the basis $\{y_1, y_2\}$ changes G by conjugation. Because in our equation L there is no first order term, we actually have that G lies in $\mathrm{SL}_2(\bar{k})$, see [**Kap76**, p. 41]. We have the following lemma, which is essentially Theorem 5.4 of [**HvdP95**].

Lemma 1.4. — *There are only three cases, with respect to G, for which k' can be different from k. These are (on an appropriate basis):*

(1) $G \subset \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \bar{k}^* \right\}, \#G > 2$, *a subgroup of a torus.*

(2) $G = D_2^{\mathrm{SL}_2}$, *a group of order* 8, *with generators* $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(3) $G = A_4^{\mathrm{SL}_2}$, *a group of order* 24.

We remark that in [**HvdP95**], the group $D_2^{\mathrm{SL}_2}$ is mistakenly denoted by D_4. We have $D_4 \neq D_2^{\mathrm{SL}_2}$, and in fact $D_2^{\mathrm{SL}_2} \cong Q_8$, where Q_8 denotes the quaternion subgroup $\{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}^*$. The notations $D_2^{\mathrm{SL}_2}$ and $A_4^{\mathrm{SL}_2}$ can be explained as follows. Using the natural homomorphism $\mathrm{SL}_2 \to \mathrm{PSL}_2$, these groups are the inverse image of $D_2 \subset \mathrm{PSL}_2$ and $A_4 \subset \mathrm{PSL}_2$ respectively. We will treat these three cases separately.

1.1. Subgroups of a torus. — In this section we consider case (1) of Lemma 1.4. There are exactly two G-invariant lines in V. These correspond to the two solutions of R_L in $\bar{k}(x)$. Such solutions are called *rational*.

For the next lemma we need to introduce the second *symmetric power* of a given differential equation. This is the differential equation with as solutions, all products of two solutions of the given equation. For example take $L : y'' = ry$, with as basis of solutions $\{y_1, y_2\}$. Then the second symmetric power of L, denoted $\mathrm{Sym}(L, 2)$ is the equation $y''' - 4ry' - 2r'y = 0$. It has $\{y_1^2, y_1y_2, y_2^2\}$ as a basis of solutions. Indeed, $\{y_1^2, y_1y_2, y_2^2\}$ are linearly independent over $\overline{k}$ (compare [**SU93**, Lemma 3.5]). In a similar way one defines higher order symmetric powers $\mathrm{Sym}(L, n)$ (see [**vdPS03**, Definition 2.24]), which we will use later on. We note that $\mathrm{Sym}(L, n)$ can have order smaller than $n + 1$. In the proof of the next lemma, we will also use that there is an action of $\mathrm{Gal}(\bar{k}/k)$ on K, which induces an action on V. It acts in the standard way on $\bar{k}(x)$. For details see [**HvdP95**].

Lemma 1.5. — *Assume we are in case* (1) *of Lemma 1.4. Then* $\mathrm{Sym}(L, 2)$ *has (up to constants) a unique non-zero solution* $H \in k(x)$*. If one of the two rational solutions of* R *does not lie in* $k(x)$*, then the rational solutions of* R *are* $\frac{H'}{2H} \pm cH^{-1}$*, for some* $c \in \overline{k} \setminus k, c^2 \in k$*.*

Proof. — For the basis $\{y_1, y_2\}$ for which the representation of G in SL_2 is as in 1. we have that y_1y_2 is G invariant, so $y_1y_2 \in \bar{k}(x)$. It is easily seen that up to constants, this is the only G-invariant solution of $\mathrm{Sym}(L, 2)$. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$ we have that $\sigma(y_1y_2)$ is another rational solution of the symmetric square, so it must be a multiple of y_1y_2. Therefore we have a $\mathrm{Gal}(\bar{k}/k)$-invariant line, and thus by Hilbert theorem 90 an invariant point on this line. After multiplying y_1 by a constant, we may suppose $H := y_1y_2 \in k(x)$. Then $\frac{H'}{H} = \frac{y_1'}{y_1} + \frac{y_2'}{y_2}$. The rational solutions of R are $\frac{y_1'}{y_1}$ and $\frac{y_2'}{y_2}$, and since $\mathrm{Gal}(\bar{k}/k)$ acts on the set of solutions of R, each one is fixed by a subgroup of $\mathrm{Gal}(\bar{k}/k)$ of index ≤ 2. Now assume this index is 2, then we can write $\frac{y_1'}{y_1} =: u =: u_0 + du_1,\ u_0, u_1 \in k(x), d^2 \in k, d \notin k$, and then $\frac{y_2'}{y_2} = u_0 - du_1$, so $\frac{H'}{H} = 2u_0$. From $u' + u^2 = r \in k(x)$ one deduces that $2u_0 = -\frac{u_1'}{u_1}$, so u_1 must be $\lambda H^{-1}, \lambda \in k^*$. Therefore we can take $c = d\lambda$, and clearly $\frac{y_2'}{y_2} = \frac{H'}{2H} - cH^{-1}$. □

We note that this gives a way to find in case (1) the rational solutions of the Riccati equation. Indeed H can be found (for example using Maple), and c can be calculated by substituting $\frac{H'}{2H} + cH^{-1}$ into the Riccati equation.

1.2. Klein's theorem. — In the remaining two cases of Lemma 1.4, the differential Galois groups are finite. This implies that the differential Galois group equals the ordinary Galois group. An important tool in studying these cases is *Klein's Theorem*. We present a version of it suggested by F. Beukers. For a different approach we refer to [**BD79**].

It will be convenient to use *differential operators*. These are elements of the skew polynomial ring $\bar{k}(x)[\partial_x]$. The multiplication is defined by $\partial_x x = x\partial_x + 1$. We will identify the linear differential equation $\Sigma_i a_i y^{(i)} = 0$ with the differential operator $\Sigma_i a_i \partial_x^i$.

We recall from [**HvdP95**] the following easy lemma.

Lemma 1.6. — *The $\bar{k}$-algebra homomorphisms $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ are given by $\phi(t) = a$ and $\phi(\partial_t) = \frac{1}{a'}\partial_x + b$ with $a \in \bar{k}(x) \setminus \bar{k}$; $a' := \frac{d}{dx}a$ and $b \in \bar{k}(x)$.*

Notation 1.7

- For $F \in \bar{k}(x)\backslash\bar{k}$ we define the $\bar{k}$-homomorphism $\phi_F : \bar{k}(t) \to \bar{k}(x)$, by $\phi_F(t) = F$.
- Let ϕ be an injective homomorphism $\phi : \bar{k}(t) \to \bar{k}(x)$. Then we also write ϕ for the extension of ϕ to the homomorphism of differential operators $\phi : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$, defined by $\phi(\partial_t) = \frac{1}{\phi(t)'}\partial_x$.
- For $F \in \overline{k}(x) \setminus \overline{k}, b \in \bar{k}(x)$, we define $\phi_{F,b} : \bar{k}(t)[\partial_t] \to \bar{k}(x)[\partial_x]$ by $\phi_{F,b}(t) = F$, $\phi_{F,b}(\partial_t) = \frac{1}{F'}(\partial_x + b)$.
- We will call an automorphism of $\overline{k}(t)[\partial_t]$, given by $t \mapsto t, \partial_t \mapsto \partial_t + b$ a *shift*.
- For a differential operator L we define $\mathrm{Aut}(L)$ to be the group $\{\psi \in \mathrm{Aut}_{\bar{k}}\,\bar{k}(t) \mid \mathrm{Norm}(\psi(L)) = L\}$.

First we will discuss the process of normalization. A second order differential operator $L := a_2\partial^2 + a_1\partial + a_0$ is said to be in *normal form* if $a_2 = 1$ and $a_1 = 0$. We can put L into normal form, $\mathrm{Norm}(L)$, by dividing L by a_2, and then applying the shift $\partial \mapsto \partial - \frac{a_1}{2a_2}$. Note that normalization transforms the old solution space V to $f \cdot V$, with $f' = \frac{a_1}{2a_2}f$. The operator remains defined over $k(x)$, but the associated Picard-Vessiot extension K changes if $f \notin K$.

Klein's theorem is concerned with differential operators $L := \partial_x^2 - r$ with finite non-cyclic differential Galois group $G \subset \mathrm{SL}_2(\bar{k})$. If we again use the notation H^{SL_2} for the inverse image in SL_2 of a group $H \subset \mathrm{PSL}_2$, the possibilities for such G are (up to conjugation): $\{D_n^{\mathrm{SL}_2}, A_4^{\mathrm{SL}_2}, S_4^{\mathrm{SL}_2}, A_5^{\mathrm{SL}_2}\}$. In [**BD79**] we find for each such group G a *standard operator*, denoted St_G, which is in normal form, and has differential Galois group G. These are:

$$\begin{aligned}
St_{D_n^{\mathrm{SL}_2}} &= \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{3}{16}\frac{1}{(t-1)^2} - \frac{n^2+2}{8n^2}\frac{1}{t(t-1)},\\
St_{A_4^{\mathrm{SL}_2}} &= \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{3}{16}\frac{1}{t(t-1)},\\
St_{S_4^{\mathrm{SL}_2}} &= \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{101}{576}\frac{1}{t(t-1)},\\
St_{A_5^{\mathrm{SL}_2}} &= \partial_t^2 + \frac{3}{16}\frac{1}{t^2} + \frac{2}{9}\frac{1}{(t-1)^2} - \frac{611}{3600}\frac{1}{t(t-1)}.
\end{aligned}$$

The so-called *local exponents* of these standard equations are given by the following table.

	0	1	∞
$St_{D_n^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{4}, \frac{3}{4}$	$-\frac{n+1}{2n}, -\frac{n-1}{2n}$
$St_{A_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{1}{3}, -\frac{2}{3}$
$St_{S_4^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{3}{8}, -\frac{5}{8}$
$St_{A_5^{\mathrm{SL}_2}}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{1}{3}, \frac{2}{3}$	$-\frac{2}{5}, -\frac{3}{5}$

In the proof of Klein's Theorem we will need the following lemma.

Lemma 1.8. — *Let L be a monic second order differential operator over $k(x)$, with finite differential Galois group G, and Picard-Vessiot extension K. Let $\{y_1, y_2\}$ be a basis of solutions of L, and write $s := \frac{y_1}{y_2}$.*

(1) *Normalizing L does not change the field $K^p := \overline{k}(x)(s) \subset K$.*

(2) *Let $L_1 \in \overline{k}(x)[\partial_x]$ be a monic differential operator, which also has a basis of solutions in K of the form $\{sy, y\}$. Then L_1 can be obtained from L by the shift $\partial_x \mapsto \partial_x - (\frac{y}{y_1})'/(\frac{y}{y_1})$.*

If moreover G is non-cyclic and $G \subset \mathrm{SL}_2(\overline{k})$, then also the following statements hold.

(3) *$K^p = K^{\pm I}$, the fixed field of $-I$ in K.*

(4) *$K = K^p(\sqrt{s'})$.*

(5) *$\overline{k}(s)$ is G-invariant and $\exists\ t \in \overline{k}(x)$ such that $\overline{k}(s)^G = \overline{k}(t)$.*

Proof

(1) This follows immediately from the fact that the normalization of L has a basis of solutions $\{fy_1, fy_2\}$ (for some f with $\frac{f'}{f} \in \overline{k}(x)$).

(2) The monic differential operator $\phi_{x,-(\frac{y}{y_1})'/(\frac{y}{y_1})}$ clearly has $\{sy, y\}$ as a basis of solutions, and therefore is equal to L_1.

(3) Since $\bar{k}(x) \subset \overline{k}(x)(y_1, y_2)$ is a finite extension, we have $y_1', y_2' \in \overline{k}(x)(y_1, y_2)$, so $K = \overline{k}(x)(y_1, y_2)$. Because K^p is algebraic over $\bar{k}(x)$ the derivation on K induces a derivation on K^p. So $(\frac{y_1}{y_2})' = \frac{d}{y_2^2} \in \bar{k}(x)(\frac{y_1}{y_2})$, where $d = y_1'y_2 - y_2'y_1$. It is easily seen that $d' = 0$, and $d \neq 0$, so $d \in \overline{k}^*$. We find that $y_2^2 \in K^p$ and for a similar reason also $y_1^2 \in K^p$. So the only elements in G that fix $\bar{k}(x)(\frac{y_1}{y_2})$ are $\pm I$. By Galois correspondence K^p is the fixed field of $\{\pm I\}$.

(4) We have $K = K^p(y_2)$, and $y_2^2 = \frac{d}{s'}$, so $K = K^p(\sqrt{s'})$.

(5) From the G-action on $\overline{k}\langle y_1, y_2\rangle$ one immediately finds that $\overline{k}(s)$ is G-invariant. Since $\bar{k}(s)$ is a purely transcendental extension of $\bar{k}$ we get by Lüroth's theorem that the fixed field of G is also purely transcendental. So we can write $\overline{k}(s)^G = \overline{k}(t)$, and because $t \in K$ is invariant under G, we get $t \in \overline{k}(x)$. ∎