Séminaires & Congrès
13, 2006, p. 313–341

HURWITZ SPACES

by

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Abstract. — This paper is intended to serve as a general introduction to the theory of Hurwitz spaces and as an overview over the different methods for their construction.

***Résumé* (Espaces de Hurwitz).** — Cet article a pour but de donner une introduction à la théorie des espaces de Hurwitz et un aperçu des différentes méthodes pour leur construction.

1. Introduction

1.1. The classical Hurwitz space and the moduli of curves. — The classical Hurwitz space first appeared in the work of Clebsch [**Cle72**] and Hurwitz [**Hur91**] as an auxiliary object to study the moduli space of curves. Let X be a smooth projective curve of genus g over $\mathbb{C}$. A rational function $f : X \to \mathbb{P}^1$ of degree n is called *simple* if there are at least $n-1$ points on X over every point of $\mathbb{P}^1$. Such a cover has exactly $r := 2g + 2n - 2$ branch points. Let $\mathcal{H}_{n,r}$ denote the set of isomorphism classes of simple branched covers of $\mathbb{P}^1$ of degree n with r branch points. Hurwitz [**Hur91**] showed that the set $\mathcal{H}_{n,r}$ has a natural structure of a complex manifold. In fact, one can realize $\mathcal{H}_{n,r}$ as a finite unramified covering

$$\Psi_{n,r} : \mathcal{H}_{n,r} \longrightarrow \mathcal{U}_r := \mathbb{P}^r - \Delta_r,$$

where Δ_r is the discriminant hypersurface. (Note that the space $\mathcal{U}_r$ has a natural interpretation as the set of all subsets of $\mathbb{P}^1$ of cardinality r. The map $\Psi_{n,r}$ sends the class of a simple cover $f : X \to \mathbb{P}^1$ to the branch locus of f.) Using a combinatorial calculation of Clebsch [**Cle72**] which describes the action of the fundamental group of $\mathcal{U}_r$ on the fibers of $\Psi_{n,r}$, Hurwitz showed that $\mathcal{H}_{n,r}$ is connected.

2000 ***Mathematics Subject Classification.*** — 14H30, 14D22.
Key words and phrases. — Covers of curves, Galois theory, Hurwitz spaces.

Later Severi [**Sev21**] proved that for $n \geq g+1$ every curve X of genus g admits a simple cover $f : X \to \mathbb{P}^1$ of degree n. In other words, the natural map

$$\mathcal{H}_{n,r} \longrightarrow \mathcal{M}_g$$

which maps the class of the cover $f : X \to \mathbb{P}^1$ to the class of the curve X is surjective. Using the connectedness of $\mathcal{H}_{n,r}$, Severi concluded that $\mathcal{M}_g$ is connected.

Although $\mathcal{M}_g$ is an algebraic variety and can be defined over $\mathbb{Z}$, the proof of its connectedness sketched above is essentially topological. It therefore does not immediately yield the connectedness of $\mathcal{M}_g \otimes \mathbb{F}_p$ for a prime p. In order to fill this gap, Fulton [**Ful69**] gave a purely algebraic construction of the Hurwitz space $\mathcal{H}_{n,r}$. In his theory, $\mathcal{H}_{n,r}$ is a scheme, of finite type over $\mathbb{Z}$, which represents a certain moduli functor. It is equipped with a natural étale morphism $\Psi_{n,r} : \mathcal{H}_{n,r} \to \mathcal{U}_r$ which becomes finite when restricted to $\operatorname{Spec} \mathbb{Z}[1/n!]$. In this setup, Fulton was able to prove the irreducibility of $\mathcal{H}_{n,r} \otimes \mathbb{F}_p$ for every prime $p > n$, using the irreducibility of $\mathcal{H}_{n,r} \otimes \mathbb{C}$. With the same reasoning as above, one can deduce the irreducibility of $\mathcal{M}_g \otimes \mathbb{F}_p$ for $p > g+1$. (At about the same time, Deligne and Mumford proved the irreducibility of $\mathcal{M}_g \otimes \mathbb{F}_p$ for all p, using much more sophisticated methods.)

Further applications of Hurwitz spaces to the moduli of curves were given by Harris and Mumford [**HM82**]. They construct a compactification $\bar{\mathcal{H}}_{n,r}$ of $\mathcal{H}_{n,r}$. Points on the boundary $\partial\bar{\mathcal{H}}_{n,r} := \bar{\mathcal{H}}_{n,r} - \mathcal{H}_{n,r}$ correspond to a certain type of degenerate covers between singular curves called *admissible covers*. The map $\mathcal{H}_{n,r} \to \mathcal{M}_g$ extends to a map $\bar{\mathcal{H}}_{n,r} \to \bar{\mathcal{M}}_g$, where $\bar{\mathcal{M}}_g$ is the Deligne–Mumford compactification of $\mathcal{M}_g$. The geometry of this map near the boundary yields interesting results on the geometry of $\bar{\mathcal{M}}_g$.

1.2. Hurwitz spaces in Galois theory. — Branched covers of the projective line have more applications besides the moduli of curves. For instance, in the context of the regular inverse Galois problem one is naturally led to study Galois covers $f : X \to \mathbb{P}^1$ with a fixed Galois group G. Here arithmetic problems play a prominent role, e.g. the determination of the minimal field of definition of a Galois cover.

Fried [**Fri77**] first pointed out that the geometry of the moduli spaces of branched covers of $\mathbb{P}^1$ with a fixed Galois group G and a fixed number of branch points carries important arithmetic information on the individual covers that are parameterized. Matzat [**Mat91**] reformulated these ideas in a field theoretic language and gave some concrete applications to the regular inverse Galois problem. Fried and Völklein [**FV91**] gave the following precise formulation of the connection between geometry and arithmetic. For a field k of characteristic 0, let $\mathcal{H}_{r,G}(k)$ denote the set of isomorphism classes of pairs (f, τ), where $f : X \to \mathbb{P}^1_k$ is a *regular* Galois cover with r branch points, defined over k, and $\tau : G \xrightarrow{\sim} \operatorname{Gal}(X/\mathbb{P}^1)$ is an isomorphism of G with the Galois group of f. Suppose for simplicity that G is center-free. Then it is proved in [**FV91**] that the set $\mathcal{H}_{r,G}(k)$ is naturally the set of k-rational points of a smooth variety $\mathcal{H}_{r,G}$, defined over $\mathbb{Q}$. Moreover, we have a finite étale cover of $\mathbb{Q}$-varieties

$$\Psi_{r,G} : \mathcal{H}_{r,G} \longrightarrow \mathcal{U}_r$$

whose associated topological covering map is determined by an explicit action of the fundamental group of $\mathcal{U}_r$ (called the *Hurwitz braid group*) on the fibres of $\Psi_{r,G}$. Using this braid action, it is shown in [**FV91**] that $\mathcal{H}_{r,G}$ has at least one absolutely irreducible component defined over $\mathbb{Q}$ if r is sufficiently large. This has interesting consequences for the structure of the absolute Galois group of $\mathbb{Q}$, see [**FV91**].

In some very special cases one can show, using the braid action on the fibres of $\Psi_{r,G}$, that $\mathcal{H}_{r,G}$ has a connected component which is a rational variety over $\mathbb{Q}$ and hence has many rational points. Then these rational points correspond to regular Galois extensions of $\mathbb{Q}(t)$ with Galois group G. For instance, [**Mat91**], §9.4, gives an example with $r = 4$ and $G = M_{24}$. This example yields the only known regular realizations of the Mathieu group M_{24}.

1.3. The general construction. — In [**FV91**] the Hurwitz space $\mathcal{H}_{r,G}$ is first constructed as a complex manifold. It is then shown to have a natural structure of a $\mathbb{Q}$-variety with the property that k-rational points on $\mathcal{H}_{r,G}$ correspond to G-Galois covers defined over k, but only for fields k of characteristic 0 (and assuming that G is center-free). From the work of Fulton one can expect that there exists a scheme $\mathcal{H}_{r,G,\mathbb{Z}}$ of finite type over $\mathbb{Z}$ such that k-rational points correspond to tamely ramified G-Galois covers over k for *all* fields k. Moreover, $\mathcal{H}_{r,G,\mathbb{Z}}$ should have good reduction at all primes p which do not divide the order of G. One can also expect that the construction of Harris and Mumford extends to the Galois situation and yields a nice compactification $\bar{\mathcal{H}}_{r,G,\mathbb{Z}}$ of $\mathcal{H}_{r,G,\mathbb{Z}}$, at least over $\mathbb{Z}[1/|G|]$. These expectations are proved in [**Wew98**], in a more general context.

If the group G has a nontrivial center, then the Hurwitz space $\mathcal{H}_{r,G,\mathbb{Z}}$ is only a *coarse* and not a *fine* moduli space. For instance, a k-rational point on $\mathcal{H}_{r,G,\mathbb{Z}}$ corresponds to a tame G-cover $f : X \to \mathbb{P}^1_{\bar{k}}$ defined over the algebraic closure of k. The field k is the *field of moduli*, but not necessarily a field of definition of f. To deal with this difficulty it is very natural to work with algebraic stacks.

The point of view of algebraic stacks has further advantages. For instance, even if G is center-free, the construction of the Harris–Mumford compactification $\bar{\mathcal{H}}_{r,G}$ of $\mathcal{H}_{r,G}$ becomes awkward without the systematic use of stacks. It also provides a much clearer understanding of the connection of Hurwitz spaces with the moduli space of curves with level structure, see [**Rom02**]. Finally, Hurwitz spaces as algebraic stacks are useful for the computation of geometric properties of the moduli of curves, e.g. Picard groups.

The present paper is intended to serve as a general introduction to the theory of Hurwitz spaces and as an overview over the different methods for their construction. For applications to arithmetic problems and Galois theory, we refer to the other contributions of this volume.

Acknowledgments. — The authors would like to thank the referee for a careful reading of a previous version of this manuscript, and in particular for pointing out a gap in the proof of Proposition 4.10.

2. Hurwitz spaces as coarse moduli spaces

In this section we define the Hurwitz space $\mathcal{H}_{r,G}$ as a coarse moduli space, using the language of schemes.

2.1. Basic definitions. — Let S be a scheme. By a *curve* over S we mean a smooth and proper morphism $X \to S$ whose (geometric) fibres are connected and 1-dimensional. If X is a curve over S, a *cover* of X is a finite, flat and surjective S-morphism $f : Y \to X$, where Y is another curve over S. We denote by $\mathrm{Aut}(f)$ the group of automorphisms of Y which leave f fixed.

A cover $f : Y \to X$ is called *Galois* if it is separable and the group $\mathrm{Aut}(f)$ acts transitively on every (geometric) fibre of f. It is called *tame* if there exists a smooth relative divisor $D \subset X$ such that the following holds: (a) the natural map $D \to S$ is finite and étale, (b) the restriction of $f : Y \to X$ to the open subset $U := X - D$ is étale, and (c) for every geometric point $s : \mathrm{Spec}\, k \to D$, the ramification index of f along D at s is > 1 and prime to the residue characteristic of s. If this is the case, the divisor D is called the *branch locus* of f. If the degree of $D \to S$ is constant and equal to r, we say that the cover f has r *branch points*.

Let G be a finite group and X a curve over S. A G-*cover* of X is a Galois cover $f : Y \to X$ together with an isomorphism $\tau : G \xrightarrow{\sim} \mathrm{Aut}(f)$. Usually we will identify the group $\mathrm{Aut}(f)$ with G.

Two G-covers $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ of the same curve X over S are called *isomorphic* if there exists an isomorphism $h : Y_1 \xrightarrow{\sim} Y_2$ such that $f_2 \circ h = f_1$ and $g \circ h = h \circ g$ for all $g \in G$.

2.2. Suppose that $S = \mathrm{Spec}\, k$, where k is a field. Then a curve X over S is uniquely determined by its function field $K := k(X)$. A cover $f : Y \to X$ corresponds one-to-one to a finite, separable and regular field extension L/K (here 'regular' means that k is algebraically closed in L). The cover f is Galois (resp. tame) if and only if the extension L/K is Galois (resp. tamely ramified at all places of K which are trivial on k).

2.3. Let us fix a finite group G and an integer $r \geq 3$. For a scheme S, we denote by $\mathbb{P}^1_S$ the relative projective line over S. Define

$$\mathcal{H}_{r,G}(S) := \{\, f : X \xrightarrow{G} \mathbb{P}^1_S \;\mid\; \deg(D/S) = r \,\}/_{\cong}$$

as the set of isomorphism classes of tame G-covers of $\mathbb{P}^1_S$ with r branch points. If $S = \operatorname{Spec} k$ then $\mathcal{H}_{r,G}$ is the set of G-Galois extensions of the rational function field $k(t)$, up to isomorphism.

2.4. The functor $S \mapsto \mathcal{H}_{r,G}(S)$ is a typical example of a *moduli problem.* One would like to show that there is a *fine moduli space* representing this functor, i.e. a scheme $\mathcal{H}$ together with an isomorphism of functors (from schemes to sets)

$$\mathcal{H}_{r,G}(S) \cong \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}).$$

Unfortunately, this is true only under an additional assumption (if and only if the group G is center free). Fortunately, one can prove a slightly weaker result without this extra assumption (see e.g. [**Wew98**]).

Theorem 2.1. — *There exists a scheme $\mathcal{H} = \mathcal{H}_{r,G,\mathbb{Z}}$, smooth and of finite type over $\mathbb{Z}$, together with a morphism of functors (from schemes to sets)*

$$\mathcal{H}_{r,G}(S) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}), \tag{1}$$

such that the following holds.

(i) *Suppose there is another scheme $\mathcal{H}'$ and a morphism of functors $\mathcal{H}_{r,G}(S) \to \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}')$. Then there exists a unique morphism of schemes $\mathcal{H} \to \mathcal{H}'$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{H}_{r,G}(S) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}) \\ & \searrow & \downarrow \\ & & \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}'). \end{array}$$

(ii) *If S is the spectrum of an algebraically closed field then* (1) *is a bijection.*

We say that the scheme $\mathcal{H} = \mathcal{H}_{r,G,\mathbb{Z}}$ is the *coarse moduli space* associated to the functor $S \mapsto \mathcal{H}_{r,G}(S)$, and call it the *Hurwitz space* for tame G-Galois covers of $\mathbb{P}^1$ with r branch points.

In particular the theorem says that for any algebraically closed field k the set $\mathcal{H}_{r,G}(k)$ (i.e. the set of isomorphism classes of regular and tamely ramified G-Galois extensions of $k(t)$) has a natural structure of a smooth k-variety $\mathcal{H}_{r,G,k}$. For k of characteristic zero this was first proved by Fried and Völklein, see [**FV91**]. In §4 we will prove it for an arbitrary field k.

Let (f, τ) be a G-cover over a scheme S. It follows immediately from the definition that the group of automorphisms of the pair (f, σ) is the center of G. It is a general fact that a coarse moduli space representing objects with no nontrivial automorphisms is actually a fine moduli space. Hence we deduce from Theorem 2.1:

Corollary 2.2. — *Suppose that the center of G is trivial. Then* (1) *is a bijection for all schemes S. In other words, the scheme $\mathcal{H}_{r,G,\mathbb{Z}}$ is a fine moduli space.*