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REMARKS TOWARDS A CLASSIFICATION OF $RS_4^2(3)$-TRANSFORMATIONS AND ALGEBRAIC SOLUTIONS OF THE SIXTH PAINLEVÉ EQUATION

by

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Abstract. — We introduce a special property, D-type, for rational functions of one variable and show that it can be effectively used for a classification of the deformations of dessins d'enfants related with the construction of algebraic solutions of the sixth Painlevé equation via the method of RS-transformations. In the framework of this classification we present a pure geometrical proof, based on the analysis of symmetry properties of the deformed dessins, of the nonexistence of some special rational coverings.

Résumé **(Remarques pour une classification des transformations de type $RS_4^2(3)$ et des solutions algébriques de la sixième équation de Painlevé)**

Nous introduisons une propriété spéciale, dite «de type D», pour les fonctions rationnelles d'une variable et nous montrons comment celle-ci pourrait être utilisée pour une classification des déformations de dessins d'enfants rattachée à la construction de solutions algébriques de l'équation de Painlevé VI via la méthode des RS-transformations. Dans le cadre de cette classification nous donnons une démonstration, purement géométrique et basée sur l'analyse des symétries des dessins déformés, de la non-existence de certains recouvrements rationnels.

1. Introduction

Recently the author introduced a general method of RS-transformations [**15**] for special functions of the isomonodromy type (SFITs) [**14**]. This method applies to SFITs defining isomonodromy deformations of linear $n \times n$-matrix ODEs of the first order with rational coefficients and with both regular and essential singular points.

RS-Transformations are just a proper combination of rational transformations (R-transformations) of the independent variable of the linear ODEs and Schlesinger transformations (S-transformation) of the dependent variable. Solutions of many different

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and seemingly unrelated problems from various areas of the theory of functions get a unified and systematic approach in the framework of this method and can be reduced to the study, construction, and classification of different RS-transformations for matrix linear ODEs.

This method, e.g., allows one to prove the duplication formula for the Gamma-function (and most probably the general multiplication formula for the multiple argument [**3**]), build higher-order transformations for the Gauss hypergeometric function and reproduce the Schwarz table for it [**2, 17**], construct quadratic transformations for the Painlevé and classical transcendental functions [**13, 16**], and provide a systematical method for finding algebraic points at which transcendental SFITs attain algebraic values [**1**]. Without doubt, many other interesting problems can be approached via the method of RS-transformations. In this paper we apply this general method to the problem of construction and classification of algebraic solutions of the sixth Painlevé equation.

Recently scanning the literature, I realized that, possibly, the first serious profound result concerning RS-transformations was obtained by F. Klein [**19**], who proved that any scalar Fuchsian equation of the second order with finite monodromy group is a "pull-back" (R-transformation) of the Euler hypergeometric equation. In this context instead of the S-transformations the notion of "projective equivalence" is used. The latter is more restrictive than general S-transformations because in terms of the matrix ODEs it corresponds to triangular Schlesinger transformations, that finally results in a more restrictive special choice of the exponent differences (formal monodromy) of the hypergeometric equation, than when more general S-transformations are allowed.

Klein's result immediately implies that any solution of the Garnier system and, in particular the sixth Painlevé equation that corresponds to a finite monodromy group of the associated Fuchsian equation, is algebraic. It is important to mention that the converse statement is not true.

In the context of the sixth Painlevé equation the first person who could, theoretically, apply the "pull-back ideology" was R. Fuchs because it was he who found that the sixth Painlevé equation governs isomonodromy deformations of the certain scalar second order Fuchsian ODE and, moreover, received an informative letter from F. Klein. He actually did it, in a study of algebraic solutions in the so-called Picard case of the sixth Painlevé equation [**10, 11**](1).

Recently appeared a paper by Ch. Doran [**8**] who formulated a more general scheme (than that used by R. Fuchs) for construction of algebraic solutions of the sixth Painlevé equation from the pull-back point of view. A more detailed account of the last work is given in Introduction of [**17**]. In the following two paragraphs we explain

(1) These works were not known to me and, possibly, to most modern researchers until very recently, when Yousuke Ohyama called our attention to them.

why the method of RS-transformations for construction of the algebraic solutions is more general than the pull-back back one.

For a given R-transformation one can normally associate a few different RS-transformations, due to the possibility of choosing different (not related by the contiguity transformations) initial hypergeometric equations, which suffer this R-transformation and, by further application of proper S-transformations, are mapped into the Fuchsian ODE with four regular points. Each of these RS-transformations generate an algebraic solution of the sixth Painlevé equation, which sometimes depends on a complex parameter. Thus we have a finite number of algebraic solutions associated with each rational function (R-transformation). On the other hand it is well known that on the set of algebraic solutions acts the subgroup of RS-transformations with $\deg R = 1$: it is just a subgroup of compositions of Möbius transformations interchanging three points 0, 1, and ∞, and those Schlesinger transformations that does not add singularities to the Fuchsian ODE with four singular points. Thus the subset of algebraic solutions associated with the same R-transformation generate a finite number of orbits of the algebraic solutions with respect to the action of the subgroup mentioned above. The minimal subset of algebraic solutions that generate these orbits are called the subset of *seed algebraic solutions*, and RS-transformations that generate them – the *seed RS-transformations*. The seed algebraic solutions corresponding to the same rational covering (R-transformation) are different, by definition; however, the seed solutions associated with different rational coverings can coincide. Furthermore, the seed solutions, even corresponding to the same rational covering, can sometimes be related by some compositions of the quadratic transformations and/or Bäcklund transformations. Since the quadratic transformations are generated by the RS-transformations with $\deg R = 2$, and one of the Bäcklund transformations has no realization as the Schlesinger transformation of the 2×2-matrix Fuchsian ODE; we call this special transformation the Okamoto transformation (see [**20**] and Appendix [**17, 18**]).

We call attention of the reader that the possibility of construction of different RS-transformations starting from the same rational covering mentioned in the previous paragraph is not considered by the successors of the “pull-back ideology” because of the projective invariance property which assumes only one particular choice of the formal monodromy of the initial hypergeometric equation. Therefore, the “pull-back results” in many cases, namely in those ones where the property of projective equivalence can be changed to a less restrictive condition of the existence of S-transformation, can be extended or completed. We discuss this opportunity for construction of higher-order transformations of the Gauss hypergeometric functions in the Remarks in Sections 4 and 5. However, it seems that the pull-back from the hypergeometric equation, due to specific properties of the hypergeometric functions, is equivalent to the formally more general method of RS-transformations. This fact we are planning to discuss in a separate paper.

This paper is a continuation of author's previous work [**17**]. In [**17**] we give a general definition of the one-dimensional deformations of dessins d'enfants and their relation to the algebraic solutions of the sixth Painlevé equation, construct by this method numerous examples of different algebraic solutions, and discuss different features of this technique, e.g., a mechanism of appearance of genus-1 algebraic solutions. In Section 2 we recall the facts from [**17**] which are necessary for understanding of this work. Here we put this technique onto a systematic footing. A new idea we use here is symmetry preserving and symmetry breaking deformations of the dessins d'enfants and their relation to uniqueness of the corresponding rational covering.

More precisely, in Section 3 we introduce a notion of the *divisor type* (D-type) of rational functions and classify all D-types of the rational functions that generate algebraic solutions of the sixth Painlevé equation via the method of RS-transformations ($R_4(3)$-*functions*). The divisor type represents a special numerical property of the critical values of rational functions, more precisely, a property of the set of multiplicities of preimages (ramification patterns) of the critical values. This set we call the type (R-type) of a rational function. Note that because of our normalization (0 and ∞ are also the critical values) a specification of the divisor type also means a special property of the divisor of zeroes and poles of our rational functions.

We call the D-*series* the set of all $R_4(3)$-functions having the same D-type. Among these D-series there are two ones with finitely many, actually a few, members. This fact is proved and the corresponding rational functions are explicitly constructed in Sections 4 and 5. Each of the other D-series, corresponding to the D-types specified in the classification theorem of Section 3, are infinite.

It is worth noticing that modern personal computers (PC) allows one to construct all rational coverings that are presented here and in [**17**] without any advanced algorithms just by the natural method explained in Remark 2.1 of [**17**]. The time of calculation with MAPLE code on a relatively powerful PC does not exceed 1 second for any of these functions. Of course, finding the concise parametrization requires much more additional time. It is interesting to note that in 1998-2000, when we used exactly the same calculational scheme but on the Pentium 2 based PC with about 256 Mb RAM, we were not able to construct many interesting functions, even some Belyi function of degree 8, see [**2**], we have found only numerically. This remark, however, does not mean that we do not need any advanced calculational algorithms; explicit construction of most of the rational coverings with the degree > 12 still represent substantial difficulties.

To each $R_4(3)$-function we also indicate the number of the seed RS-transformations and present one algebraic solution whose construction does not require explicit form of the related Schlesinger transformation. It is exactly the "pull-back" solution, to get explicitly the other seed solutions one has to construct (explicitly) corresponding S-transformations. This procedure is absolutely straightforward and does not require any advance computer algorithms and we do not consider it here. Numerous examples of the complete constructions of RS-transformations are given in [**1**].

This paper is a far-going extension of the second part of my talk in Angers, where I have only explained some simplest ideas concerning the concept of deformations of the dessins d'enfants and announced the construction of the solution presented in Section 4.

In the proofs of sections 4 and 5 we substantially use a graphical representation of the rational functions introduced in [**17**], which we call the *deformation dessins*. The reader should consult this work for a better understanding of these proofs, however I hope that the general idea and the scheme of these proofs can be understood even with the help of the following comments. In case, $R_4(3)$-function exists there is at least one graph, constructed according the rules given in [**17**], which represents it. In the proofs of nonexistence of some rational functions we use the evident fact that if the graph (the deformation dessin) does not exist, then clearly the rational function does not exist. In case some deformation dessin exists, it defines R-type, the conjecture, which is made in [**17**], says, that in this case rational function also exists. So, the statement, of *existence* of certain rational mappings which is based on existence of the deformation dessins is conventional and assumes the validity of this conjecture. In fact, for all rational functions, which existence we claim, we give either explicit formulae, or prove that they can be presented as the composition of explicitly known functions. So, all our proofs of existence of rational functions are based on explicit constructions and therefore also does not rely on any hypothesis.

Every deformation dessin can be obtained from a proper Grothendieck's dessin d'enfant as a result of the so-called *face deformations*: the *join* and *cross*. We consider also one more face deformation which is called the *twist*, however the latter can be treated as a special case of the join. We also consider vertex deformations, however, they can be avoided, more precisely instead we can always consider proper face deformations of an equivalent rational function transformed under a proper the Möbius transforms.

Suppose that for a given R-type there exists a corresponding rational function. Such rational function normally is not unique. Say, rational functions corresponding to $R_4(3)$-types often depends on one (sometimes on a few (!)) additional parameters. Moreover, there always exists a parametrization of these functions that they become rational functions of these additional parameters. Clearly, the latter parametrization is not unique: we can make Möbius transformations of the independent variable of our rational function with the coefficients depending on the additional parameters and also substitutions of the additional parameters by rational functions of additional parameters. However, even modulo such transformations the rational functions are not uniquely defined by their R-types. Some light on this problem is brought by the