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ISOMONODROMY FOR COMPLEX LINEAR q-DIFFERENCE EQUATIONS

by

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Anyone who considers transcendental
means of producing Galois groups
is, of course, in a state of sin. (*)

Abstract. — The words "monodromy" and "isomonodromy" are used in the theory of difference and q-difference equations by Baranovsky-Ginzburg, Jimbo-Sakai, Borodin, Krichever,... although it is not clear that phenomena of branching during analytic continuation are involved there. In order to clarify what is at stake, we survey results obtained during the last few years, mostly by J.-P. Ramis, J. Sauloy and C. Zhang. Links to Galois theory (as developped by P. Etingof, M. van der Put & M. Singer, Y. André, L. Di Vizio...) are briefly mentioned. A tentative definition of isomonodromy deformations is given along with some elementary results.

***Résumé* (Isomonodromie des équations aux q-différences complexes).** — Les mots « monodromie« et « isomonodromie » ont été employés en théorie des équations aux différences et aux q-différences par Baranovsky-Ginzburg, Jimbo-Sakai, Borodin, Krichever,... bien que, dans un tel contexte, n'apparaissent pas clairement des phénomènes de ramification par prolongement analytique. Afin de clarifier ce qui est en jeu, nous décrivons des résultats obtenus ces dernières années, principalement par J.-P. Ramis, J. Sauloy et C. Zhang. Les liens avec la théorie de Galois (telle qu'elle a été développée par P. Etingof, M. van der Put & M. Singer, Y. André, L. Di Vizio...) sont brièvement mentionnés. Une définition expérimentale de déformation isomonodromique est proposée, ainsi que quelques résultats élémentaires.

0. Introduction

0.1. Roots. — In recent years, the words "monodromy", "isomonodromy" have been used in various places in the context of difference and q-difference equations, *e.g.*, see V. Baranovsky & V. Ginzburg ([**6**]), M. Jimbo & H. Sakai, drawing on previous results

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(*) Adapted from von Neumann who said "*arithmetical* means of producing *random digits*", see [**41**].

of Jimbo and Miwa ([**21**]), A. Borodin ([**10**]) and, more recently(1), I. Krichever ([**22**]). However, in these contexts, it is not clear that problems of multivalued solutions and branching at singularities are really involved, as it is the case in the classical setting of linear differential equations in the complex plane. The goal of this survey, is to summarize what can be said about an underlying geometry or topology of solutions encoded in a monodromy group or a Galois group, even if the solutions are taken to be uniform. We shall stick to q-differences, since the theory looks much better behaved there than for differences. Moreover, we shall almost only refer to work conducted under the impulse of Jean-Pierre Ramis, mostly by J.-P. Ramis, C. Zhang and the author. Note that this is meant to be a survey paper: essentially no proofs are given. On the other hand, for a survey with a broader scope, [**14**] is recommended.

While the prehistory of q-difference equations may be thought to have started with Euler, the archetypal example certainly is Heine's *basic hypergeometric series*, here written for a "base" $q \in \mathbf{C}$ such that $|q| > 1$ (see [**14**]):

$$\Phi(a,b,c;q,z) = \sum_{n\geq 0} \frac{(a;p)_n (b;p)_n}{(c;p)_n (p;p)_n} z^n, \qquad \text{where } p = q^{-1} \text{ and } (x;p)_n = \prod_{i=0}^{n-1} (1 - xp^i).$$

It is a q-analogue of the Gauss hypergeometric series

$$F(\alpha,\beta,\gamma;z) = \sum_{n\geq 0} \frac{(\alpha)_n (\beta)_n}{(1)_n (\gamma)_n} z^n, \qquad \text{where } (\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i).$$

The most obvious analogy is that, if one takes $a = p^\alpha$, $b = p^\beta$, $c = p^\gamma$ and lets q go to 1, then, the coefficients of the series defining $\Phi(a,b,c;q,z)$ tend to the coefficients of the series defining $F(\alpha,\beta,\gamma;z)$.

A deeper analogy is related to functional equations. The function $\Phi = \Phi(a,b,c;q,z)$ is solution of a second order linear q-difference equation with rational coefficients, that is, it satisfies a $\mathbf{C}(z)$-linear relation on $\Phi(z)$, $\Phi(qz)$ and $\Phi(q^2 z)$. This relation can be written in terms of the operator σ_q defined by $\sigma_q \phi(z) = \phi(qz)$, thus giving rise to the relation

$$(0.0.1) \qquad \sigma_q^2 \Phi - \lambda \sigma_q \Phi + \mu \Phi = 0 \quad \text{with} \begin{cases} \lambda = \dfrac{(a+b)z - (1 + c/q)}{abz - c/q} \\ \mu = \dfrac{z - 1}{abz - c/q} \end{cases}.$$

(1) The ArXiv preprint by Krichever appeared one month after the Painlevé conference for which this talk was prepared.

It can also be given in terms of the operator δ_q defined by $\delta_q\phi(z) = \dfrac{\phi(qz)-\phi(z)}{q-1}$. It, then, takes the form

$$\delta_q^2\Phi - \tilde{\lambda}(q)\delta_q\Phi + \tilde{\mu}(q)\Phi = 0 \quad \text{with} \begin{cases} \tilde{\lambda}(q) = \dfrac{\lambda-2}{q-1} \\ \tilde{\mu}(q) = \dfrac{\mu-\lambda+1}{(q-1)^2} \end{cases}. \tag{0.0.2}$$

If one brutally (or heuristically) replaces the operator δ_q by the Euler differential operator $\delta = z\, d/dz$, and the coefficients by their limit as q goes to 1, one finds the corresponding *hypergeometric differential equation* satisfied by $F = F(\alpha,\beta,\gamma;z)$:

$$\delta^2 F - \tilde{\lambda}\delta F + \tilde{\mu}F = 0 \quad \text{with} \begin{cases} \tilde{\lambda} = \dfrac{(\alpha+\beta)z+(1-\gamma)}{1-z} \\ \tilde{\mu} = \dfrac{\alpha\beta z}{1-z} \end{cases}. \tag{0.0.3}$$

Since this equation was the first instance of the so-called Riemann-Hilbert correspondence, one would expect this limiting process to be reflected on the monodromy: the general theory shall be mentioned in 1.3, this particular example being dealt with, in full detail, in [**35**]. We shall rather use the operator σ_q and also rather use systems than equations. For instance, putting $X = \begin{pmatrix} f \\ \sigma_q f \end{pmatrix}$, we get the system

$$\sigma_q X = AX \quad \text{with } A = \begin{pmatrix} 0 & 1 \\ -\mu & \lambda \end{pmatrix} \in GL_2(\mathbf{C}(z)). \tag{0.0.4}$$

The modern history begins with the famous paper by Birkhoff about the so-called generalized Riemann problem, [**8**]. There, he tackled what we perhaps would call nowadays the Riemann-Hilbert problem of classifying differential equations by their singularities and (loosely said) global geometric behaviour. For regular singular differential equations, the former are encoded in the local Jordan structure (generically, the eigenvalues or *exponents*) and the latter means the monodromy representation or, in less intrinsic terms, the knowledge of sufficiently many connection matrices. Birkhoff showed that, to a large extent, the problem can be posed and solved in parallel for differential, difference and q-difference equations. For definiteness, from now on, we consider (as Birkhoff did) q-difference systems meromorphic over the Riemann sphere. To be more precise, we first introduce some notations.

Throughout the text, q is a fixed complex number with modulus $|q| > 1$[(2)]. We also fix a $\tau \in \mathcal{H}$ (the Poincaré half plane) such that $q = e^{-2\imath\pi\tau}$. The field K of

(2) The opposite convention (that is, $0 < |q| < 1$) holds equally often in the litterature, for instance in [**18**]; some formulas or definitions (e.g., classical "basic" functions or the Newton polygon) do depend on the chosen convention. The fundamental fact, if one wants to do some analysis, is that $|q| \neq 1$ (at least in the present state of our technology).

coefficients is one of the following: the field $\mathbf{C}(z)$ of rational functions (*global* case), the field $\mathbf{C}(\{z\})$ of convergent Laurent series meromorphic at 0 (*analytic* or *convergent* local case) and the field $\mathbf{C}((z))$ of formal Laurent series meromorphic at 0 (*formal* local case); we understand meromorphic Laurent series to have finitely many negative exponents. Any of these fields can be endowed with an automorphism σ_q defined by $(\sigma_q f)(z) = f(qz)$. A linear q-difference equation of order n may be written

$$\sigma_q^n(f) + a_1\sigma_q^{n-1}(f) + \cdots + a_n f = 0\ ,\ a_1, \ldots, a_n \in K\ ,\ a_n \neq 0. \tag{0.0.5}$$

By vectorializing, *i.e.*, setting

$$X = X_f \underset{\text{def}}{=} \begin{pmatrix} f \\ \sigma_q f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix}$$

$$\text{and} \qquad A = A_{\underline{a}} \underset{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{pmatrix}, \tag{0.0.6}$$

the equation (0.0.5) may be turned into a system

$$\sigma_q X = AX\ ,\ A \in GL_n(K). \tag{0.0.7}$$

For any such equation or system with coefficients in the field K, one will look for solutions in some K-algebra of functions $\mathcal{A}$ endowed with a dilatation operator σ_q extending the one of K. One possible choice for $\mathcal{A}$ is the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions over $\mathbf{C}^*$, with the natural operation defined by $(\sigma_q f)(z) = f(qz)$. The subalgebra of *q-constants*

$$\mathcal{A}^{\sigma_q} \underset{def}{=} \{f \in \mathcal{A} \ /\ \sigma_q f = f\}$$

is then the field $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ of q-invariant meromorphic functions. Letting $z = e^{2\imath\pi x}$, we see that $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ is isomorphic to the field of meromorphic functions over $\mathbf{C}$ with periods 1 and τ, thus, to a field of elliptic functions. More geometrically, $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ can be identified in a natural way to the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. The latter is an elliptic curve, since the exponential map $x \mapsto e^{2\imath\pi x}$ makes $\mathbf{C}$ a covering of $\mathbf{C}^*$ and induces an isomorphism $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) \to \mathbf{E}_q$.

If we find a fundamental solution of (0.0.7) in $\mathcal{A} = \mathcal{M}(\mathbf{C}^*)$, that is, a matrix $\mathcal{X} \in GL_n(\mathcal{A})$ such that $\sigma_q\mathcal{X} = A\mathcal{X}$, then the vector solutions $X \in \mathcal{A}^n$ are exactly the vectors $\mathcal{X}C$ with $C \in (\mathcal{A}^{\sigma_q})^n$, that is, they form a vector space of rank n over the field of constants $\mathcal{M}(\mathbf{E}_q)$. As a matter of fact, contrary to the case of differential equations, a fundamental solution in $\mathcal{A} = \mathcal{M}(\mathbf{C}^*)$ always exists: one does not have to rely on multivalued functions (but, see remark 0.1 below).

Birkhoff considered systems (0.0.7) for $K = \mathbf{C}(z)$. These should be classified with respect to rational equivalence:

$$B \sim A \iff B = F[A] \underset{def}{=} (\sigma_q F) A F^{-1} \text{ for an } F \in GL_n(\mathbf{C}(z)). \tag{0.0.8}$$

Note that the gauge transformation $X \mapsto Y = FX$ changes solutions of the system $\sigma_q X = AX$ into solutions of the system $\sigma_q Y = BY$. By the way, Birkhoff proved that any such system is equivalent to the system obtained through (0.0.6) from some equation (0.0.5), a result known today as the *cyclic vector lemma*.

To begin with, the matrix F has coefficients in the field $K = \mathbf{C}(z)$ (global classification); but intermediate results involve local classification, for which we allow local gauge transforms $F \in GL_n(\mathbf{C}(\{z\}))$ or $F \in GL_n(\mathbf{C}((z)))$. In this setting, the only possible *local* information seems to be located at 0 and ∞, since they are the only points fixed by the automorphism $z \mapsto qz$. Birkhoff (relying on previous results by Adams and Carmichael) then defined what it means for a system to be singular regular at these points and built multivalued local solutions at 0 and ∞. The *a priori* local solutions $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ thus obtained are actually meromorphic all over $\mathbf{C}^*$, because the functional equation $\sigma_q X = AX$ (A rational) expands any given disk of convergence by the factor $|q| > 1$.

Then, solutions $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ being given, he defines their connection matrix P through the relation: $\mathcal{X}^{(0)} = \mathcal{X}^{(\infty)} P$. Since $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(\infty)}$ are fundamental solutions of the same q-difference system, the matrix P is q-invariant, thus elliptic: it can therefore be encoded by finitely many numerical invariants. These, of course, should be joined with the local invariants at 0 and ∞ (the exponents).

In order to compare the class of q-difference systems (up to rational equivalence) to the class of such sets of invariants (up to natural symetries), Birkhoff counted the number of free parameters on both sides and found them equal. Then, he formulated the inverse problem in the generic case (the local matrices $A(0)$ and $A(\infty)$ are semisimple): does every such family of numerical invariants come from a regular singular system ? He solved this "generalized Riemann problem" affirmatively. Here, the main tool was the "preliminary theorem", better known as "Birkhoff factorization of matrices". Nowadays, it is rather formulated as the Birkhoff-Grothendieck theorem about the classification of holomorphic vector bundles over the Riemann sphere (see [**5**] or [**25**]), making it quite clear that it has a topological meaning. Besides, this theorem was used in this form by Röhrl in [**31**] to solve the Riemann-Hilbert problem for differential equations (see also [**32**]).