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ON A LOCAL REDUCTION OF A HIGHER ORDER PAINLEVÉ EQUATION AND ITS UNDERLYING LAX PAIR NEAR A SIMPLE TURNING POINT OF THE FIRST KIND

by

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Abstract. — We discuss a local reduction theorem for 0-parameter solutions of a higher order Painlevé equation and its underlying Lax pair near a simple turning point of the first kind when the size of the Lax pair is greater than 2. As a typical example of such higher order Painlevé equations the Noumi-Yamada systems are mainly considered.

***Résumé* (Sur une réduction locale au voisinage d'un point tournant simple de première espèce des équations de Painlevé d'ordre supérieur et de leur paire de Lax)**

Nous considérons les solutions sans paramètre d'une équation de Painlevé d'ordre supérieur au voisinage d'un point tournant simple et sa paire de Lax associée. Nous présentons un théorème de réduction locale et nous développons comme cas typique l'exemple des systèmes de Noumi-Yamada.

1. Introduction

The local reduction theorem for 0-parameter solutions of the traditional (i.e., second order) Painlevé equations with a large parameter (cf. [**3**], see also [**5**]) is generalized to those of some higher order Painlevé equations in [**6**] (cf. [**4**] for its announcement). That is, it is shown in [**6**] that a 0-parameter solution of each member of the first and second Painlevé hierarchies $(P_J)_m$ (J = I, II-1 and II-2; $m = 1, 2, 3, \dots$) discussed in [**2**] can be locally reduced to a 0-parameter solution of the traditional first Painlevé equation

$$(P_{\mathrm{I}}) \qquad \frac{d^2u}{dt^2} = \eta^2(6u^2 + t)$$

near a simple turning point of $(P_J)_m$ of the first kind in the sense of [**2**]. In [**6**], to construct a local transformation which reduces a 0-parameter solution of $(P_J)_m$ to

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that of $(P_{\rm I})$, we make essential use of the fact that the Lax pair $(L_J)_m$ associated with $(P_J)_m$ consists of 2×2 systems. The purpose of this paper is to discuss the local reduction theorem for 0-parameter solutions of higher order Painlevé equations near a simple turning point of the first kind in the case where the size of the underlying Lax pair is greater than 2.

In this paper, as an example of higher order Painlevé equations whose underlying Lax pair is of size greater than 2, we mainly discuss the Noumi-Yamada systems [**7**], i.e., higher order Painlevé equations with the affine Weyl group symmetry of type $A_l^{(1)}$ $(l=2,3,4,\dots)$. The Noumi-Yamada systems can be considered as higher order analogue of the traditional fourth and fifth Painlevé equations $(P_{\rm IV})$ and $(P_{\rm V})$. As the size of the Lax pair associated with the Noumi-Yamada system of type $A_l^{(1)}$ is $l+1$, the result of [**6**] is not applicable to this case; instead we construct the reduction of the underlying Lax pair of the Noumi-Yamada system to that of the traditional first Painlevé equation $(P_{\rm I})$. This means that the local reduction for a 0-parameter solution of the Noumi-Yamada system is also constructed implicitly. For the precise statement of our main theorem see Theorem 2.2 in Section 2.

The plan of the paper is as follows: After recalling the explicit form of the Noumi-Yamada systems and reviewing some basic properties of their Stokes geometry studied in [**9**], we state our main theorem in Section 2. To prove our main theorem, we construct two reductions of the underlying Lax pair of the Noumi-Yamada system to that of $(P_{\rm I})$ by the medium of the local reduction of a pair of first order linear systems to its normal form at a (simple or double) turning point discussed in [**8**], and employ a kind of "matching" method for the two reductions thus constructed. In Section 3 we briefly explain the results of [**8**] necessary for the proof of our main theorem and study the structure of transformations which keep the normal form at a turning point invariant. Using these results and a matching method, we finally give a proof of our main theorem in Section 4.

2. Main result

To state our main theorem we need to prepare some notions and notations about the Noumi-Yamada system and its Stokes geometry. Let us first recall the explicit form of the Noumi-Yamada system and its underlying Lax pair.

The Noumi-Yamada system of type $A_l^{(1)}$ in case l is even (i.e., when $l=2m$; $m=1,2,\dots$) is the following system of first order nonlinear differential equations:

$$\frac{du_j}{dt}=\eta\Big[u_j(u_{j+1}-u_{j+2}+\cdots-u_{j+2m})+\alpha_j\Big] \tag{1}$$

$(j=0,1,\dots,2m)$, where α_j are complex parameters satisfying

$$\alpha_0+\cdots+\alpha_{2m}=\eta^{-1} \tag{2}$$

and the unknown functions u_j and the independent variable t are normalized so that

$$u_0 + \cdots + u_{2m} = t \tag{3}$$

may be satisfied, while in case l is odd (i.e., when $l = 2m+1$; $m = 1, 2, \dots$) it is given by

$$\frac{t}{2}\frac{du_j}{dt} = \eta\Big[u_j\Big(\sum_{1\le r\le s\le m} u_{j-1+2r}u_{j+2s} - \sum_{1\le r\le s\le m} u_{j+2r}u_{j+1+2s}\Big) + \frac{t}{2}\alpha_j\Big] \tag{4}$$

($j = 0, 1, \dots, 2m+1$), where α_j, u_j and t satisfy the following:

$$\alpha_0 + \alpha_2 + \cdots + \alpha_{2m} = \alpha_1 + \alpha_3 + \cdots + \alpha_{2m+1} = \eta^{-1}/2, \tag{5}$$

$$u_0 + u_2 + \cdots + u_{2m} = u_1 + u_3 + \cdots + u_{2m+1} = t/2. \tag{6}$$

The Lax pair associated with the Noumi-Yamada system of type $A_l^{(1)}$ consists of the following first order $N \times N$ ($N = l + 1$) systems of linear differential equations:

$$\frac{\partial}{\partial x}\psi = \eta A\psi, \quad \frac{\partial}{\partial t}\psi = \eta B\psi, \tag{7}$$

where

$$A = -\frac{1}{x}\begin{pmatrix} \epsilon_1 & u_1 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \epsilon_{N-2} & u_{N-2} & 1 \\ x & & & \epsilon_{N-1} & u_{N-1} \\ xu_0 & x & & & \epsilon_N \end{pmatrix} \tag{8}$$

and

$$B = \begin{pmatrix} q_1 & -1 & & & \\ & q_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & q_{N-1} & -1 \\ -x & & & & q_N \end{pmatrix}. \tag{9}$$

That is, (1) (resp., (4)) describes the compatibility condition

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + \eta(AB - BA) = 0 \tag{10}$$

of (7) for $l = 2m$, i.e., $N = 2m + 1$ (resp., for $l = 2m + 1$, i.e., $N = 2m + 2$). Here ϵ_j are parameters determined by the relations $\alpha_j = \epsilon_j - \epsilon_{j+1} + \eta^{-1}\delta_{j,0}$ and $\epsilon_1 + \cdots + \epsilon_N = 0$ ($\delta_{j,k}$ stands for Kronecker's delta), and $q_j = q_j(t)$ are functions of t satisfying $q_{j+2} - q_j = u_j - u_{j+1}$ and $q_1 + \cdots + q_N = -t/2$.

As (1) is equivalent to the traditional fourth Painlevé equation (P_{IV}) when $l = 2$ (i.e., $m = 1$), Equation (1) can be considered as a higher order fourth Painlevé equation; Equation (1) and its underlying Lax pair (7) for $l = 2m$ are respectively referred to as $(P_{\mathrm{IV}})_m$ and $(L_{\mathrm{IV}})_m$ in what follows. Similarly Equation (4) and its underlying Lax pair (7) for $l = 2m + 1$ are respectively referred to as $(P_{\mathrm{V}})_m$ and

$(L_{\rm V})_m$, as (4) is equivalent to the traditional fifth Painlevé equation $(P_{\rm V})$ when $l=3$ (i.e., $m=1$).

Our problem is to analyze the Noumi-Yamada system and its underlying Lax pair near a simple turning point of the first kind. A turning point of the Noumi-Yamada system and its basic properties are studied in [**9**]. It is defined as a turning point of the linearized equation ("Fréchet derivative") at a 0-parameter solution. Here a 0-parameter solution of the Noumi-Yamada system is a formal solution of the form

$$\widehat{u}_j = \widehat{u}_j(t,\eta) = \widehat{u}_{j,0}(t) + \eta^{-1}\widehat{u}_{j,1}(t) + \cdots \tag{11}$$

($0 \le j \le 2m$ for $(P_{\rm IV})_m$ and $0 \le j \le 2m+1$ for $(P_{\rm V})_m$), and the linearized equation at $\widehat{u} = \{\widehat{u}_j\}$ is an equation obtained by setting $u_j = \widehat{u}_j + \Delta u_j$ in $(P_{\rm IV})_m$ or $(P_{\rm V})_m$ and by taking its linear part in $\{\Delta u_j\}$. Note that the linearized equation at a 0-parameter solution $\{\widehat{u}_j\}$ is a system of first order linear differential equations for $\Delta u = {}^t(\Delta u_0, \ldots, \Delta u_l)$ ($l = 2m$ for $(P_{\rm IV})_m$ and $l = 2m+1$ for $(P_{\rm V})_m$) and can be expressed as

$$\frac{d}{dt}\Delta u = \eta C \Delta u, \quad C = C(t,\eta) = C_0(t) + \eta^{-1}C_1(t) + \cdots . \tag{12}$$

A turning point of the first kind of the Noumi-Yamada system is then, by definition, a point $t=\tau$ where two non-trivial solutions $\nu^{\pm}(t)$ of the characteristic equation $\det(\nu - C_0(t)) = 0$ of (12) merge and their values $\nu^{\pm}(\tau)$ are equal to 0. That is, if we let P denote a polynomial of ν defined by $\nu^{-1}\det(\nu - C_0(t))$ for $(P_{\rm IV})_m$ and by $\nu^{-2}\det(\nu - C_0(t))$ for $(P_{\rm V})_m$ (cf. [**9**, Proposition 2.3]), a turning point of the first kind is a point $t=\tau$ where $\nu=0$ is a double root of $P=0$. Note that a turning point of the first kind is also a branch point of the Riemann surface $\mathcal{R}$ associated with the 0-parameter solution. In what follows we assume that a turning point $t=\tau$ of the first kind is a square-root type branch point of $\mathcal{R}$ and that τ is simple in the sense of [**1**]; to be more specific, using a local parameter $s = (t-\tau)^{1/2}$ of $\mathcal{R}$ at $t=\tau$, we require that the polynomial $P = P(s,\nu)$ of ν should satisfy the following conditions at $(s,\nu) = (0,0)$:

$$P(0,0) = \frac{\partial P}{\partial \nu}(0,0) = 0, \quad \frac{\partial P}{\partial s}(0,0) \neq 0, \quad \frac{\partial^2 P}{\partial \nu^2}(0,0) \neq 0. \tag{13}$$

Substituting a 0-parameter solution $\{\widehat{u}_j\}$ of the Noumi-Yamada system into the coefficients of the underlying Lax pair (7), we now obtain the Lax pair

$$\frac{\partial}{\partial x}\psi = \eta A \psi, \quad A = A(x,t,\eta) = A_0(x,t) + \eta^{-1}A_1(x,t) + \cdots , \tag{14}$$

$$\frac{\partial}{\partial t}\psi = \eta B \psi, \quad B = B(x,t,\eta) = B_0(x,t) + \eta^{-1}B_1(x,t) + \cdots , \tag{15}$$

the compatibility condition of which is satisfied as a formal power series of η^{-1}. Then, as is proved in [**9**, Theorem 2.1], a double turning point $x = b(t)$ of the first equation (14) of the Lax pair merges with a simple turning point $x = a(t)$ of (14) at a turning point $t=\tau$ of the first kind of the Noumi-Yamada system, provided that the following genericity condition should hold at $x = a(t)$, which is also a turning point of the second equation (15) of the Lax pair:

(16) At $x = a(t)$ exactly two eigenvalues of $B_0(x, t)$ merge and the other eigenvalues are mutually distinct.

Note that the same pair of the eigenvalues of $A_0(x, t)$, denoted by $\lambda^{\pm}(x, t)$, merges both at $x = b(t)$ and at $x = a(t)$. Furthermore, letting $\nu^{\pm}(t)$ denote the two non-trivial solutions of the characteristic equation $\det(\nu - C_0(t)) = 0$ of (12) satisfying $\nu^+(\tau) = \nu^-(\tau) = 0$ and $\nu^-(t) = -\nu^+(t)$, we find that the following relation holds:

$$\frac{1}{2}\int_{\tau}^{t}(\nu^+(t) - \nu^-(t))dt = \int_{a(t)}^{b(t)}(\lambda^+(x, t) - \lambda^-(x, t))dx. \tag{17}$$

This relation (17) guarantees that, if $t = \sigma$ is a point on a Stokes curve of the Noumi-Yamada system emanating from τ, i.e., a curve in the t-plane (or, rather on the Riemann surface $\mathcal{R}$) given by

$$\operatorname{Im}\int_{\tau}^{t}(\nu^+(t) - \nu^-(t))dt = 0, \tag{18}$$

and further if $t = \sigma$ is sufficiently close to τ, then the two turning points $b(\sigma)$ and $a(\sigma)$ of (14) are connected by a Stokes curve (or, rather Stokes segment) of (14). The Stokes segment of (14), denoted by $\gamma = \gamma(\sigma)$, connecting $b(\sigma)$ and $a(\sigma)$ plays a crucially important role in the following argument; we try to construct a transformation which reduces the Lax pair (14) and (15) of the Noumi-Yamada system to that of the traditional first Painlevé equation (P_{I}) semi-globally near γ.

In view of (16), as the same pair $\lambda^{\pm}(x, t)$ of the eigenvalues of A_0 merges both at $x = b(t)$ and at $x = a(t)$, the Lax pair (14) and (15) can be simultaneously block-diagonalized in a neighborhood of $(x, t) = (a(\tau), \tau)$ $(= (b(\tau), \tau))$. (For the block-diagonalization we refer the reader to, e.g., [**8**, Proposition 1]. See also [**10**], [**11**].) That is, (14) and (15) can be transformed into a system of the form

$$\frac{\partial}{\partial x}\tilde{\psi} = \eta\tilde{A}(x, t, \eta)\tilde{\psi}, \quad \tilde{A}(x, t, \eta) = \left(\begin{array}{c|c} A^{(1)} & 0 \\ \hline 0 & A^{(2)} \end{array}\right), \tag{19}$$

$$\frac{\partial}{\partial t}\tilde{\psi} = \eta\tilde{B}(x, t, \eta)\tilde{\psi}, \quad \tilde{B}(x, t, \eta) = \left(\begin{array}{c|c} B^{(1)} & 0 \\ \hline 0 & B^{(2)} \end{array}\right), \tag{20}$$

where $A^{(1)} = \sum_j \eta^{-j} A_j^{(1)}$ and $B^{(1)} = \sum_j \eta^{-j} B_j^{(1)}$ are (formal power series of η^{-1} with coefficients of) 2×2 matrices while $A^{(2)}$ and $B^{(2)}$ are $(l-1)\times(l-1)$ diagonal matrices with distinct diagonal components, by a transformation $\psi = (\sum_j \eta^{-j} P_j(x, t))\tilde{\psi}$ in a neighborhood of $(x, t) = (a(\tau), \tau)$ (in particular, in a neighborhood of the Stokes segment γ). Here the eigenvalues of $A_0^{(1)}$ are given by the merging ones $\lambda^{\pm}(x, t)$ and hence the problem is reduced to that for the 2×2 blocks,