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THE ELLIPTIC REPRESENTATION OF THE SIXTH PAINLEVÉ EQUATION

by

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Abstract. — We find a class of solutions of the sixth Painlevé equation corresponding to *almost all* the monodromy data of the associated linear system; actually, all data but one point in the space of data. We describe the critical behavior close to the critical points by means of the elliptic representation, and we find the relation among the parameters at the different critical points (connection problem).

***Résumé* (Représentation elliptique de l'équation de Painlevé VI).** — Nous exhibons une classe de solutions de l'équation de Painlevé VI prenant en compte *presque toutes* les données de monodromie du système linéaire associé ; en fait, toutes les données sauf un point de l'espace des données de monodromie.

Nous décrivons le comportement critique au voisinage de chaque point critique au moyen de la représentation elliptique. Nous explicitons les relations liant les paramètres aux différents points critiques (problème de connexion).

1. Introduction

In this paper, I review some results [**6, 7**] on the elliptic representation of the general Painlevé 6 equation (PVI in the following). I would like to explain the motivations which brought me to study the elliptic representation, and the problems which such an approach has solved.

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The sixth Painlevé equation is

$$\text{(PVI)}\quad \frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right].$$

The generic solution has essential singularities and/or branch points in 0,1,∞. These points will be called *critical.* The other singularities, which depend on the initial conditions, are poles. The behavior of a solution close to a critical point is called *critical behavior*. A solution of PVI can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. For generic values of the integration constants and of the parameters α,β,γ,δ, it can not be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent.*

The first analytical problem with Painlevé equations is to determine the critical behavior of the transcendents at the critical points. Such a behavior must depend on two parameters (integration constants). The second problem, called *connection problem*, is to find the relation between the couples of parameters at different critical points.

2. Previous Results

The work of Jimbo [**9**] is the fundamental paper on the subject. For generic values of α, β, γ δ, PVI admits a 2-parameter class of solutions, with the following critical behavior: .

$$y(x)=a^{(0)}x^{1-\sigma^{(0)}}(1+O(|x|^{\epsilon})),\qquad x\to 0, \tag{1}$$

$$y(x)=1-a^{(1)}(1-x)^{1-\sigma^{(1)}}(1+O(|1-x|^{\epsilon})),\qquad x\to 1, \tag{2}$$

$$y(x)=a^{(\infty)}x^{\sigma^{(\infty)}}(1+O(|x|^{-\epsilon})),\qquad x\to\infty, \tag{3}$$

where ϵ is a small positive number, $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)}\neq 0$ and

$$0\le\Re\sigma^{(i)}<1. \tag{4}$$

We remark that x converges to the critical points *inside a sector* with vertex on the corresponding critical point. The connection problem is to finding the relation among the three pairs $(\sigma^{(i)},a^{(i)})$, $i=0,1,\infty$. In [**9**] the problem is solved by the isomonodromy deformations theory. Actually, PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations [**12, 10, 11**]

$$\frac{dY}{dz}=A(z;x)Y,\qquad A(z;x):=\left[\frac{A_0(x)}{z}+\frac{A_x(x)}{z-x}+\frac{A_1(x)}{z-1}\right].$$

The 2×2 matrices $A_i(x)$ ($i = 0, x, 1$ are labels) depend on x in such a way that the monodromy of a fundamental solution $Y(z,x)$ does not change for small deformations of x. They also depend on the parameters $\alpha, \beta, \gamma, \delta$ of PVI. Here, we use the same notations of the paper [**9**]: namely, $A_0(x) + A_1(x) + A_x(x) = -\frac{1}{2}\mathrm{diag}(\theta_\infty, -\theta_\infty)$; the eigenvalues of $A_i(x)$ are $\pm\frac{1}{2}\theta_i$, $i = 0, 1, x$, and

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_x^2. \tag{5}$$

The equations of monodromy-preserving deformation (Schlesinger equations), can be written in Hamiltonian form [**15**] and reduce to PVI, being the transcendent $y(x)$ solution of $A(y(x);x)_{1,2} = 0$.

Let M_0, M_1, M_x be the monodromy matrices at $z = 0, 1, x$, for a given basis in the fundamental group of $\mathbf{P}^1 \backslash \{0, 1, x, \infty\}$.There is a one to one correspondence(1) between a given choice of *monodromy data* θ_0, θ_x, θ_1, θ_∞, $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_0M_1)$, $\mathrm{tr}(M_1M_x)$ and a transcendent $y(x)$ (see [**9, 2, 6**]) . Namely:

$$y(x) = y\big(x;\, \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big). \tag{6}$$

We remark that $\theta_0, \theta_x, \theta_1, \theta_\infty$ specify the equation. Only two of $\mathrm{tr}(M_0M_x)$, $\mathrm{tr}(M_0M_1)$, $\mathrm{tr}(M_1M_x)$ are independent, because, for a given choice of the basis of loops in $\mathbf{P}^1 \backslash \{0, 1, x, \infty\}$, we have $M_\infty = M_1M_xM_0$. This implies

$$\begin{aligned}\cos(\pi\theta_0)\mathrm{tr}(M_1M_x) + \cos(\pi\theta_1)\mathrm{tr}(M_0M_x) + \cos(\pi\theta_x)\mathrm{tr}(M_1M_0)& \\ = 2\cos(\pi\theta_\infty) + 4\cos(\pi\theta_1)\cos(\pi\theta_0)\cos(\pi\theta_x)&.\end{aligned}$$

A transcendent in the class (1) (2) (3) above, coincides with a transcendent (6), for:

$$\begin{aligned} 2\cos(\pi\sigma^{(0)}) &= \mathrm{tr}(M_0M_x), \\ 2\cos(\pi\sigma^{(1)}) &= \mathrm{tr}(M_1M_x), \\ 2\cos(\pi\sigma^{(\infty)}) &= \mathrm{tr}(M_0M_1) \end{aligned} \tag{7}$$

and

$$a^{(i)} = a^{(i)}\big(\sigma^{(i)}; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1), \mathrm{tr}(M_1M_x)\big), \qquad i = 0, 1, \infty. \tag{8}$$

Formula (8) for $a^{(0)}$, can be derived from (1.8), (1.10) and (2.15) of [**9**](2). It can be derived also from (A.6), (A.28), (A.29) of [**7**] (note that in [**7**] I miss-printed (A.30),

(1)If $\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}$.

(2)The connection problem is solved in [**9**] for generic values of α, β, γ, δ . More precisely, by *generic case* we mean:

$$\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}; \qquad \frac{\pm\sigma^{(i)} \pm \theta_1 \pm \theta_\infty}{2},\ \frac{\pm\sigma^{(i)} \pm \theta_0 \pm \theta_x}{2}\ \notin \mathbf{Z}. \tag{9}$$

The signs $\pm$ vary independently. This is a technical condition which can be abandoned. For example, the non-generic case $\beta = \gamma = 1 - 2\delta = 0$ and α any complex number was analyzed in [**2**], for its relevant applications to Frobenius manifolds. Its elliptic representation is discussed in [**6**].

which can be anyway corrected using (A.28), (A.29). Also in formula (1.8) of [**9**] there is a miss-print, I think: the last sign is $\pm$ and not $\mp$.).

(10)
$$a^{(0)}=\frac{1}{4}\frac{[(\theta_x+\sigma^{(0)})^2-\theta_0^2][\theta_\infty+\theta_1+\sigma^{(0)}]}{{\sigma^{(0)}}^2[\theta_\infty+\theta_1-\sigma^{(0)}]}$$
$$\times\frac{\Gamma(1+\sigma^{(0)})^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x-\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0-\sigma^{(0)})+1\right)}{\Gamma(1-\sigma^{(0)})^2\Gamma\left(\frac{1}{2}(\theta_0+\theta_x+\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_x-\theta_0+\sigma^{(0)})+1\right)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1-\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty-\sigma^{(0)})+1\right)}{\Gamma\left(\frac{1}{2}(\theta_\infty+\theta_1+\sigma^{(0)})+1\right)\Gamma\left(\frac{1}{2}(\theta_1-\theta_\infty+\sigma^{(0)})+1\right)}\times\frac{V}{U}$$
$$U:=\left[\frac{i}{2}\sin(\pi\sigma^{(0)})\mathrm{tr}(M_1M_x)-\cos(\pi\theta_x)\cos(\pi\theta_\infty)-\cos(\pi\theta_0)\cos(\pi\theta_1)\right]e^{i\pi\sigma^{(0)}}$$
$$+\frac{i}{2}\sin(\pi\sigma^{(0)})\mathrm{tr}(M_0M_1)+\cos(\pi\theta_x)\cos(\pi\theta_1)+\cos(\pi\theta_\infty)\cos(\pi\theta_0)$$
$$V:=4\sin\frac{\pi}{2}(\theta_0+\theta_x-\sigma^{(0)})\sin\frac{\pi}{2}(\theta_0-\theta_x+\sigma^{(0)})$$
$$\times\sin\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma^{(0)}))\sin\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma^{(0)}).$$

The formulas of $a^{(1)}$, $a^{(\infty)}$, are given in Remark 2 below. The monodromy data are restricted by the following condition, equivalent to (4):

$$\mathrm{tr}(M_iM_j)\notin(-\infty,-2],\qquad j=0,1,x. \tag{11}$$

I take the occasion to say that in [**7**] the condition (1.30) is wrong, the right one being (11).

Remark 1. — PVI depends holomorphically on θ_0, θ_1, θ_x, θ_∞; and so does $y(x)$. On the other hand, the matrices of the Fuchsian system have a pole in $\theta_\infty=0$. This is a non-generic case, which must be treated separately. The non-generic cases have been studied, for the equation with $\theta_0=\theta_x=\theta_1=0$ and arbitrary θ_∞. The reader is referred to [**14, 2, 6**]. Also in these cases, $y(x)$ is shown to depend holomorphically on θ_∞[(3)].

We also remark that formula (10) is to be modified when $\sigma^{(0)}=0$. We refer to [**9**].

(3) From the technical point of view, one has to solve a Riemann-Hilbert problem, to construct the fuchsian system associated to PVI from the given set of monodromy data. If θ_∞ is not integer, the monodromy at infinity is similar to the matrix $\mathrm{diag}(e^{-i\pi\theta_\infty}, e^{i\pi\theta_\infty})$. But if the condition $\theta_\infty\in\mathbf{Z}$ is broken, the monodromy contains non diagonal terms. The solution of the problem is possible case by case, and it is reduced to a connection problem for hyper-geometric equations with logarithmic solutions and non-generic monodromy.

Remark 2. — To describe the symmetries of PVI, it may be convenient to choose

$$\alpha = \frac{1}{2}\theta_\infty^2. \tag{12}$$

PVI is invariant for the change of variables $y(x) = 1-\tilde{y}(t)$, $x = 1-t$ and simultaneous permutation of θ_0, θ_1. This means that $y(x)$ solves PVI if and only if $\tilde{y}(t)$ solves PVI with permuted parameters and independent variable t. Similarly, PVI is invariant for $y(x) = 1/\tilde{y}(t)$, $x = 1/t$ and simultaneous permutation of θ_∞, θ_0. It is invariant for $y(x) = (\tilde{y}(t) - t)/(1 - t)$, $x = t/(t - 1)$ and simultaneous permutation of θ_0, θ_x. By composing the third, first and again third symmetries, we get $y(x) = \tilde{y}(t)/t$, $t = 1/x$ with the permutation of θ_1, θ_x. Therefore, the critical points $0, 1, \infty$ are equivalent. This means that it is enough to know (8) for $a^{(0)}$, to write the analogous for $a^{(1)}$ and $a^{(\infty)}$. Explicitly, to compute $a^{(1)}$ one has to do the following substitution in the formula of $a^{(0)}$:

$$\sigma \mapsto \sigma^{(1)}$$

$$\theta_0 \mapsto \theta_1, \quad \theta_1 \mapsto \theta_0 \tag{13}$$

$$\mathrm{tr}(M_0M_x) \mapsto \mathrm{tr}(M_1M_x), \quad \mathrm{tr}(M_1M_x) \mapsto \mathrm{tr}(M_0M_x), \tag{14}$$

$$\mathrm{tr}(M_0M_1) \mapsto 4\left[\cos(\pi\theta_0)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_x)\right] + \\ - \left(\mathrm{tr}(M_0M_1) + \mathrm{tr}(M_0M_x)\mathrm{tr}(M_1M_x)\right) \tag{15}$$

to compute $a^{(\infty)}$ one has to do the following substitution in the formula of $a^{(0)}$:

$$\sigma \mapsto \sigma^{(\infty)}$$

$$\theta_x \mapsto \theta_1, \quad \theta_1 \mapsto \theta_x \tag{16}$$

$$\mathrm{tr}(M_0M_x) \mapsto \mathrm{tr}(M_0M_1), \tag{17}$$

$$\mathrm{tr}(M_0M_1) \mapsto 4\big[\cos(\pi\theta_x)\cos(\pi\theta_0) + \cos(\pi\theta_\infty)\cos(\pi\theta_1)\big] + \\ - \left(\mathrm{tr}(M_0M_x) + \mathrm{tr}(M_1M_x)\mathrm{tr}(M_0M_1)\right). \tag{18}$$

In the above formula we used the definition (5) for θ_∞.

3. Two Questions

Problem 1. — Let PVI be given; namely, let θ_0, θ_1, θ_x, θ_∞ be given. We would like to study *all* the solutions of the given PVI. As a consequence of the one-to-one correspondence (6) between monodromy data and transcendents, we need to compute the critical behavior and solve the connection problem for all values $\mathrm{tr}(M_iM_j)$, $j = 0, 1, x$[4].

This problem was for me the first motivation to study the elliptic representation.

[4] In exceptional cases (θ_0, θ_x, θ_1, $\theta_\infty \in \mathbf{Z}$) the one-to-one correspondence is broken. They can be treated separately. See for example [**14**].