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# STABILITY OF QUANTUM HARMONIC OSCILLATOR UNDER TIME QUASI-PERIODIC PERTURBATION

*by*

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***Abstract.*** — We prove stability of the bound states for the quantum harmonic oscillator under non-resonant, time quasi-periodic perturbations by proving that the associated Floquet Hamiltonian has pure point spectrum.

***Résumé*** **(Stabilité de l'oscillateur harmonique quantique sous les perturbations quasi-périodiques)**

Nous démontrons la stabilité des états bornés de l'oscillateur harmonique sous les perturbations non-résonantes, quasi-périodiques en temps en démontrant que l'hamiltonien Floquet associé a un spectre purement ponctuel.

The stability of the quantum harmonic oscillator is a long standing problem since the establishment of quantum mechanics. The Schrödinger equation for the harmonic oscillator in $\mathbb{R}^n$ (in appropriate coordinates) is the following:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}\sum_{i=1}^{n}(-\frac{\partial^2}{\partial x_i^2}+x_i^2)\psi, \tag{1}$$

where we assume

$$\psi \in C^1(\mathbb{R}, L^2(\mathbb{R}^n)) \tag{2}$$

for the moment. We start from the 1 dimensional case, $n=1$. (1) then reduces to

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2}+x^2)\psi. \tag{3}$$

The Schrödinger operator

$$H = \frac{1}{2}(-\frac{d^2}{dx^2}+x^2) \tag{4}$$

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is the 1-d harmonic oscillator. Since $H$ is independent of $t$, it is amenable to a spectral analysis. It is well known that $H$ has pure point spectrum with eigenvalues

$$\lambda_n = 2n+1, \quad n = 0, 1..., \tag{5}$$

and eigenfunctions (the Hermite functions)

$$h_n(x) = \frac{H_n(x)}{\sqrt{2^n n!}} e^{-x^2/2}, \quad n = 0, 1... \tag{6}$$

where $H_n(x)$ is the $n^{\text{th}}$ Hermite polynomial, relative to the weight $e^{-x^2}$ ($H_0(x) = 1$) and

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \sqrt{\pi}\delta_{mn} \tag{7}$$

Using (5-7), the normalized $L^2$ solutions to (1) are all of the form

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n h_n(x) e^{i\frac{\lambda_n}{2}t} \quad (\sum |a_n|^2 = 1), \tag{8}$$

corresponding to the initial condition

$$\psi(x,0) = \sum_{n=0}^{\infty} a_n h_n(x) \quad (\sum |a_n|^2 = 1). \tag{9}$$

The functions in (8) are almost-periodic (in fact periodic here) in time with frequencies $\lambda_n/4\pi$, $n = 0, 1...$

Equation (3) generates a unitary propagator $U(t,s) = U(t-s,0)$ on $L^2(\mathbb{R})$. Since the spectrum of $H$ is pure point, $\forall u \in L^2(\mathbb{R})$, $\forall \epsilon$, $\exists R$, such that

$$\inf_t \|U(t,0)u\|_{L^2(|x|\le R)} \ge (1-\epsilon)\|u\| \tag{10}$$

by using eigenfunction (Hermite function) expansions. The harmonic oscillator (4) is an integrable system. The above results are classical. It is natural to ask how much of the above picture remains under perturbation, when the system is no longer integrable. In this paper, we investigate stability of the 1-d harmonic oscillator under time quasi-periodic, spatially localized perturbations. To simplify the exposition, we study the following "model" equation:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta|h_0(x)|^2 \sum_{k=1}^{\nu} \cos(\omega_k t + \phi_k)\psi, \tag{11}$$

on $C^1(\mathbb{R}, L^2(\mathbb{R}))$, where

$$0 < \delta \ll 1,\ \omega = \{\omega_k\}_{k=1}^{\nu} \in [0,2\pi)^{\nu},\ \phi = \{\phi_k\}_{k=1}^{\nu} \in [0,2\pi)^{\nu},\ h_0(x) = e^{-x^2/2}. \tag{12}$$

In particular, we shall study the validity of (10) for solutions to (11), when $U$ is the propagator for (11). The method used here can be generalized to treat the equation

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta V(t,x), \tag{13}$$

where $V$ is $C_0^{\infty}$ in $x$ and analytic, quasi-periodic in $t$.

The perturbation term, $O(\delta)$ term in (11) is motivated by the nonlinear equation:

$$-i\frac{\partial}{\partial t}\psi = \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + M\psi + \delta|\psi|^2\psi \quad (0 < \delta \ll 1), \tag{14}$$

where $M$ is a Hermite multiplier, *i.e.*, in the Hermite function basis,

$$M = \operatorname{diag}(M_n), \quad M_n \in \mathbb{R}, \tag{15}$$

$$Mu = \sum_{n=0}^{\infty} M_n(h_n, u)h_n, \text{ for all } u \in L^2(\mathbb{R}). \tag{16}$$

Specifically, (11) is motivated by the construction of time quasi-periodic solutions to (14) for appropriate initial conditions such as

$$\psi(x, 0) = \sum_{i=1}^{\nu} c_{k_i} h_{k_i}(x). \tag{17}$$

In (11), for computational simplicity, we take the spatial dependence to be $|h_0(x)|^2$ as it already captures the essence of the perturbation in view of (14, 17, 6). The various computations and the Theorem extend immediately to more general finite combinations of $h_k(x)$.

*The Floquet Hamiltonian and formulation of stability.* — It follows from [**32, 33**] that (11) generates a unique unitary propagator $U(t, s)$, $t$, $s \in \mathbb{R}$ on $L^2(\mathbb{R})$, so that for every $s \in \mathbb{R}$ and

$$u_0 \in H^2 = \{f \in L^2(\mathbb{R}) | \|f\|_{H^2}^2 = \sum_{|\alpha+\beta|\leq 2} \|x^\alpha \partial_x^\beta f\|_{L^2}^2 < \infty\}, \tag{18}$$

$$u(\cdot) = U(\cdot, s)u_0 \in C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C^0(\mathbb{R}, H^2) \tag{19}$$

is a unique solution of (11) in $L^2(\mathbb{R})$ satisfying $u(s) = u_0$.

When $\nu = 1$, (11) is time periodic with period $T = 2\pi/\omega$. The 1-period propagator $U(T + s, s)$ is called the Floquet operator. The long time behavior of the solutions to (11) can be characterized by means of the spectral properties of $U(T + s, s)$ [**14, 21, 34**]. Furthermore the nature of the spectrum of $U$ is the same (apart from multiplicity) as that of the Floquet Hamiltonian $K$ [**31**]:

$$K = i\omega\frac{\partial}{\partial \phi} + \frac{1}{2}(-\frac{\partial^2}{\partial x^2} + x^2)\psi + \delta|h_0(x)|^2 \cos\phi \tag{20}$$

on $L^2(\mathbb{R}) \otimes L^2(\mathbb{T})$, where $L^2(\mathbb{T})$ is $L^2[0, 2\pi)$ with periodic boundary conditions.

Decompose $L^2(\mathbb{R})$ into the pure point $H_{pp}$ and continuous $H_c$ spectral subspaces of the Floquet operator $U(T + s, s)$:

$$L^2(\mathbb{R}) = H_{pp} \oplus H_c. \tag{21}$$

We have the following equivalence relations [**14, 34**]: $u \in H_{pp}(U(T + s, s))$ if and only if $\forall\epsilon > 0$, $\exists R > 0$, such that

$$\inf_t \|U(t, s)u\|_{L^2(|x|\leq R)} \geq (1 - \epsilon)\|u\|; \tag{22}$$

and $u \in H_c(U(T+s,s))$ if and only if $\forall R>0$,

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t dt'\|U(t',s)u\|^2_{L^2(|x|\le R)}=0. \tag{23}$$

(Needless to say, the above statements hold for general time periodic Schrödinger equations.)

When $\nu \geq 2$, (10) is time quasi-periodic. The above constructions extend for small $\delta$, cf. [**1, 12, 22**] leading to the Floquet Hamiltonian $K$:

$$K = i\sum_{k=1}^{\nu}\omega_k\frac{\partial}{\partial\phi_k}+\frac{1}{2}(-\frac{\partial^2}{\partial x^2}+x^2)\psi+\delta|h_0(x)|^2\sum_{k=1}^{\nu}\cos\phi_k \tag{24}$$

on $L^2(\mathbb{R})\otimes L^2(\mathbb{T}^\nu)$, cf. [**7**]. This is related to the so called reducibility of skew product flows in dynamical systems, cf. [**12**]. We note that the Hermite-Fourier functions:

$$e^{-in\cdot\phi}h_j(x),\quad n\in\mathbb{Z}^\nu,\quad \phi\in\mathbb{T}^\nu,\quad j\in\{0,1...\} \tag{25}$$

provide a basis for $L^2(\mathbb{R})\otimes L^2(\mathbb{T}^\nu)$.

We say that the harmonic oscillator $H$ is *stable* if $K$ has pure point spectrum. Let $s\in\mathbb{R}$. This implies (by expansion using eigenfunctions of $K$) that given any $u\in L^2(\mathbb{R})$, $\forall\epsilon>0$, $\exists R>0$, such that

$$\inf_t\|U(t,s)u\|_{L^2(|x|\le R)}\geq(1-\epsilon)\|u\|, \text{a.e.}\,\phi, \tag{26}$$

cf. [**7, 22**]. So (10) remains valid and we have dynamical stability. We now state the main results pertaining to (11).

**Theorem.** — *There exists $\delta_0>0$, such that for all $0<\delta<\delta_0$, there exists $\Omega\subset[0,2\pi)^\nu$ of positive measure, asymptotically full measure:*

$$\text{mes}\,\Omega\to(2\pi)^\nu\quad \text{as } \delta\to 0, \tag{27}$$

*such that for all $\omega\in\Omega$, the Floquet Hamiltonian $K$ defined in (24) has pure point spectrum: $\sigma(K)=\sigma_{pp}$. Moreover the Fourier-Hermite coefficients of the eigenfunctions of $K$ have subexponential decay.*

As an immediate consequence, we have

**Corollary.** — *Assume that $\Omega$ is as in the Theorem. Let $s\in\mathbb{R}$. For all $\omega\in\Omega$, all $u\in L^2(\mathbb{R})$, all $\epsilon>0$, there exists $R>0$, such that*

$$\inf_t\|U(t,s,\phi)u\|_{L^2(|x|\le R)}\geq(1-\epsilon)\|u\|, \text{a.e.}\,\phi, \tag{28}$$

*where $U$ is the unitary propagator for (11).*

We note that this good set $\Omega$ of $\omega$ is a subset of Diophantine frequencies. This is typical for KAM type of persistence theorem. Stability under time quasi-periodic perturbations as in (11) is, generally speaking a precursor for stability under nonlinear perturbation as in (14) (cf. [**7, 6**]), where $M$ plays the role of $\omega$ and varies the tangential frequencies. The above Theorem resolves the Enss-Veselic conjecture dated from their 1983 paper [**14**] in a general quasi-periodic setting.

*A sketch of the proof of the Theorem.* — Instead of working with $K$ defined on $L^2(\mathbb{R})\otimes L^2(\mathbb{T}^\nu)$ directly, it is more convenient to work with its unitary equivalent $H$ on $\ell^2(\mathbb{Z}^\nu\times\{0,\,1...\})$, using the Hermite-Fourier basis in (25). We have

$$H = \text{diag}\ (n\cdot\omega + j + \frac{1}{2}) + \frac{\delta}{2} W\otimes\Delta \tag{29}$$

on $\ell^2(\mathbb{Z}^\nu\times\{0,\,1...\})$, where $W$ acts on the $j$ indices, $j=0,\,1,\,2...$,

$$W_{jj'}\sim\frac{1}{\sqrt{j+j'}}e^{-\frac{(j-j')^2}{2(j+j')}}\quad\text{for } j+j'\gg 1; \tag{30}$$

$\Delta$ acts on the $n$ indices, $n\in\mathbb{Z}^\nu$,

$$\Delta_{nn'}=1,\quad |n-n'|_{\ell^1}=1,\ \Delta_{nn'}=0,\quad\text{otherwise.} \tag{31}$$

The computation of $W$ involves integrals of products of Hermite functions. We will explain shortly this computation, which is independent from the main thread of construction.

The principal new feature here is that $W$ is long range. The $j^{\text{th}}$ row has width $O(\sqrt{j})$ about the diagonal element $W_{jj}$. It is *not* and *cannot* be approximated by a convolution matrix. The potential $x^2$ breaks translational invariance. The annihilation and creation operators of the harmonic oscillator $a=\frac{1}{\sqrt{2}}(\frac{d}{dx}+x)$, $a^*=\frac{1}{\sqrt{2}}(-\frac{d}{dx}+x)$, satisfying $[a,a^*]=1$, are generators of the Heisenberg group. So (19) presents a new class of problems distinct from that considered in [**2, 3, 4, 7, 6, 13, 24, 26**].

The proof of pure point spectrum of $H$ is via proving pointwise decay of the finite volume Green's functions: $(H_\Lambda-E)^{-1}$, where $\Lambda$ are finite subsets of $\mathbb{Z}^\nu\times\{0,\,1...\}$ and $\Lambda\nearrow\mathbb{Z}^\nu\times\{0,\,1...\}$. We need decay of the Green's functions at all scales, as assuming $E$ an eigenvalue, *a priori* we do not have information on the center and support of its eigenfunction $\psi$. The regions $\Lambda$ where $(H_\Lambda-E)^{-1}$ has pointwise decay is precisely where we establish later that $\psi$ is small there.

For the initial scales, the estimates on $G_\Lambda(E)=(H_\Lambda-E)^{-1}$ are obtained by direct perturbation theory in $\delta$ for $0<\delta\ll 1$. For subsequent scales, the proof is a multiscale induction process using the resolvent equation. Assume we have estimates on $G_{\Lambda'}$ for cubes $\Lambda'$ at scale $L'$. Assume $\Lambda$ is a cube at a larger scale $L$, $L\gg L'$. Intuitively, if we could establish that for most of $\Lambda'\subset\Lambda$, $G_{\Lambda'}(E)$ has pointwise decay, then assuming we have some *a priori* estimates on $G_\Lambda(E)$, we should be able to prove that $G_\Lambda(E)$ also has pointwise decay.

There are "two" directions in the problem, the higher harmonics direction $n$ and the spatial direction $j$. The off-diagonal part of $H$ is Toeplitz in the $n$ direction, corresponding to the discrete Laplacian $\Delta$. Since the frequency $\omega$ is in general a vector (if $\nu\geq 2$), $n\cdot\omega$ does not necessarily $\to\infty$ as $|n|\to\infty$. So the $n$ direction is non-perturbative. We use estimates on $G_{\Lambda'}$ and semi-algebraic techniques as in [**5, 7**] to control the number of resonant $\Lambda'$, where $G_{\Lambda'}$ is large, in $\Lambda$.