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FOUR LAMBDA STORIES, AN INTRODUCTION TO COMPLETELY INTEGRABLE SYSTEMS

by

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Abstract. — Among all non-linear differential equations arising in Physics or in geometry, completely integrable systems are exceptional cases, at the concurrence of miraculous symmetry properties. This text proposes an introduction to this subject, through a list of examples (the sinh-Gordon, Toda, Korteweg-de Vries equations, the harmonic maps, the anti-self-dual connections on the four-dimensional space). The leading thread is the parameter lambda, which governs the algebraic structure of each of these systems.

Résumé **(Quatre histoires de lambda, une introduction aux systèmes complètement intégrables)**

Parmi toutes les équations différentielles non linéaires venant de la physique ou de la géométrie, les systèmes complètement intégrables sont des cas exceptionnels, où se conjuguent des propriétés de symétries miraculeuses. Ce texte propose une introduction à ce sujet, à travers une liste d'exemples (les équations de sh-Gordon, de Toda, de Korteweg-de Vries, les applications harmoniques, les connexions anti-auto-duales sur l'espace de dimension quatre). Le fil conducteur est le paramètre lambda, qui gouverne la structure algébrique de chacun de ces systèmes.

Introduction

Completely integrable systems are non linear differential equations or systems of differential equations which possess so much symmetry that it is possible to construct by quadratures their solutions. But they have something more: in fact the appellation 'completely integrable' helps to summarize a concurrence of miraculous properties which occur in some exceptional situations. Some of these properties are: a Hamiltonian structure, with as many conserved quantities and symmetries as the number of degrees of freedom, the action of Lie groups or, more generally, of affine Lie algebras, a reformulation of the problem by a *Lax equation*. One should also add

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that, in the best cases, these non linear equations are converted into linear ones after a transformation which is more or less the Abel map from a Riemann surface to a Jacobian variety, and so on. Each one of these properties captures an essential feature of completely integrable systems, but not the whole picture.

Hence giving a complete and concise definition of an integrable system seems to be a difficult task. And moreover the list of known completely integrable systems is quite rich today but certainly still not definitive. So in this introduction text I will just try to present different examples of such systems, some are ordinary differential equations, the other ones are partial differential equations from physics or from differential geometry. I will unfortunately neglect many fundamental aspects of the theory (such as the spectral curves, the R-matrix formulation and its relation to quantum groups, the use of symplectic reduction, etc.) and privilege one point of view: in each of these examples a particular *character*, whose presence was not expected at the beginning, appears and plays a key role in the whole story. Although the stories are very different you will recognize this *character* immediately: *his* name is λ and *he* is a complex parameter.

In the first section we outline the Hamiltonian structure of completely integrable systems and expound the Liouville–Arnold theorem. In the second section we introduce the notion of *Lax equation* and use ideas from the Adler–Kostant–Symes theory to study in details the Liouville equation $\frac{d^2}{dt^2}q + 4e^{2q} = 0$ and an example of the Toda lattice equation. We end this section by a general presentation of the Adler–Kostant–Symes theory. Then in the third section, by looking at the sinh–Gordon equation $\frac{d^2}{dt^2}q + 2\sinh(2q) = 0$, we will meet for the first time λ: here this parameter is introduced *ad hoc* in order to converte infinite dimensional matrices to finite dimensional matrices depending on λ.

The second λ story is about the KdV equation $\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} = 0$ coming from fluid mechanics. There λ comes as the eigenvalue of some auxiliary differential operator involved in the Lax formulation and hence is often called the *spectral parameter*. We will see also how the Lax equation can be translated into a zero-curvature condition. A large part of this section is devoted to a description of the Grassmannian of G. Segal and G. Wilson and of the τ-function of M. Sato and may serve for instance as an introduction before reading the paper by Segal and Wilson [**29**].

The third λ story concerns constant mean curvature surfaces and harmonic maps into the unit sphere. Although the discovery of the completely integrable structure of these problems goes back to 1976 [**27**], λ was already observed during the ninetenth century by O. Bonnet [**7**] and is related somehow to the existence of conjugate families of constant mean curvature surfaces, a well-known concept in the theory of minimal surfaces through the Weierstrass representation. This section is relatively short since the Author already wrote a monograph on this subject [**18**] (see also [**17**]).

The fourth λ story is part of the twistor theory developped by R. Penrose and his group during the last 40 years. The aim of this theory was initially to understand relativistic partial differential equations like the Einstein equation of gravity and the Yang–Mills equations for gauge theory in dimension 4, through complex geometry. Eventually this theory had also application to elliptic analogues of these problems on Riemannian four-dimensional manifolds. Here λ has also a geometrical flavor. If we work with a Minkowski metric then λ parametrizes the light cone directions or the celestial sphere through the stereographic projection. In the Euclidean setting λ parametrizes complex structures on a 4-dimensional Euclidean space. Here we will mainly focus on anti-self-dual Yang–Mills connections and on the Euclidean version of Ward's theorem which characterizes these connections in terms of holomorphic bundles.

A last general remark about the meaning of λ is that for all equations with Lax matrices which are polynomial in λ, the characteristic polynomial of the Lax matrix defines an algebraic curve, called the *spectral curve*, and λ is then a coordinate on this algebraic curve. Under some assumptions (e.g. for *finite gap* solutions of the KdV equation or for *finite type* harmonic maps) the Lax equation linearizes on the Jacobian of this algebraic curve.

The Author hopes that after reading this text the reader will feel the strong similarities between all these different examples. It turns out that these relationships can be precised, this is for instance the subject of the books [**22**] or [**21**]. Again the aim of this text is to present a short introduction to the subject to non specialists having a basic background in analysis and differential geometry. The interested reader may consult [**10**], [**13**], [**14**], [**17**], [**19**], [**23**] [**24**], [**32**] for more refined presentations and further references.

1. Finite dimensional integrable systems: the Hamiltonian point of view

Let us consider the space $\mathbb{R}^{2n}$ with the coordinates $(q,p) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$. Many problems in Mechanics (and in other branches of mathematical science) can be expressed as the study of the evolution of a point in such a space, governed by the **Hamilton system of equations**

$$\left\{ \begin{array}{rcl} \dfrac{dq^i}{dt} & = & \dfrac{\partial H}{\partial p_i}(q(t),p(t)) \\ \dfrac{dp_i}{dt} & = & -\dfrac{\partial H}{\partial q^i}(q(t),p(t)), \end{array} \right.$$

where we are given a function $H : \mathbb{R}^{2n} \longmapsto \mathbb{R}$ called **Hamiltonian function**.

For instance paths $x : [a,b] \longrightarrow \mathbb{R}^3$ which are solutions of the Newton equation $m\ddot{x}(t) = -\nabla V(x(t))$ are critical points of the Lagrangian functional

$\mathcal{L}[x] := \int_a^b [\frac{m}{2}|\dot{x}(t)|^2 - V(x(t))]dt$. And by the Legendre transform they are converted into solutions of the Hamilton system of equations in $(\mathbb{R}^6, \omega)$ for $H(q,p) := \frac{|p|^2}{2m} + V(q)$.

We can view this system of equations as the flow of the **Hamiltonian vector field** defined on $\mathbb{R}^{2n}$ by

$$\xi_H(q,p) := \sum_i \frac{\partial H}{\partial p_i}(q,p)\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}(q,p)\frac{\partial}{\partial p_i}.$$

A geometrical, coordinate free, characterization of ξ_H can be given by introducing the **canonical symplectic form** on $\mathbb{R}^{2n}$

$$\omega := \sum_{i=1}^n dp_i \wedge dq^i.$$

Indeed ξ_H is the unique vector field which satisfies the relations

$$\forall (q,p), \mathbb{R}^{2n}, \forall X = \sum_i V^i \frac{\partial}{\partial q^i} + W_i \frac{\partial}{\partial p_i}, \quad \omega_{(q,p)}(\xi_H(q,p), X) + dH_{(q,p)}(X) = 0.$$

A notation is convenient here: given a vector $\xi \in \mathbb{R}^{2n}$ and for any $(q,p) \in \mathbb{R}^{2n}$, we denote by $\xi \lrcorner \omega_{(q,p)}$ the 1-form defined by $\forall X \in \mathbb{R}^{2n}$, $\xi \lrcorner \omega_{(q,p)}(X) = \omega_{(q,p)}(\xi, X)$. Then the preceding relation is just that $\xi_H \lrcorner \omega + dH = 0$ everywhere.

We call $(\mathbb{R}^{2n}, \omega)$ a **symplectic space**. More generally, given a smooth manifold $\mathcal{M}$, a *symplectic form* ω on $\mathcal{M}$ is a 2-form such that: (i) ω *is closed*, i.e., $d\omega = 0$, and (ii) ω *is non degenerate*, i.e., $\forall x \in \mathcal{M}$, $\forall \xi \in T_x\mathcal{M}$, if $\xi \lrcorner \omega_x = 0$, then $\xi = 0$. Note that the property (ii) implies that the dimension of $\mathcal{M}$ must be even. Then $(\mathcal{M}, \omega)$ is called a **symplectic manifold**.

1.1. The Poisson bracket. — We just have seen a rule which associates to each smooth function $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ a vector field ξ_f (i.e., such that $\xi_f \lrcorner \omega + df = 0$). Furthermore for any pair of functions $f, g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ we can define a third function called the **Poisson bracket** of f and g

$$\{f, g\} := \omega(\xi_f, \xi_g).$$

One can check easily that

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}.$$

In classical (i.e., not quantum) Mechanics the Poisson bracket is important because of the following properties:

1. if $\gamma = (q,p) : [a,b] \longrightarrow \mathbb{R}^{2n}$ is a solution of the Hamilton system of equations with the Hamiltonian H and if $f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is a smooth function, then

$$\frac{d}{dt}\left(f(\gamma(t))\right) = \{H, f\}(\gamma(t)).$$

This can be proved by a direct computation, either in coordinates:

$$\begin{aligned}\frac{d}{dt}(f\circ\gamma) &= \sum_i \frac{\partial f}{\partial p_i}(\gamma)\frac{dp_i}{dt} + \frac{\partial f}{\partial q^i}(\gamma)\frac{dq^i}{dt}\\ &= \sum_i \frac{\partial f}{\partial p_i}(\gamma)\left(-\frac{\partial H}{\partial q^i}(\gamma)\right) + \frac{\partial f}{\partial q^i}(\gamma)\left(\frac{\partial H}{\partial p_i}(\gamma)\right)\\ &= \{H, f\}\circ\gamma.\end{aligned}$$

or by a more intrinsic calculation:

$$\frac{d}{dt}(f\circ\gamma) = df_\gamma(\dot\gamma) = df_\gamma(\xi_H(\gamma)) = -\omega_\gamma(\xi_f(\gamma), \xi_H(\gamma)) = \{H, f\}\circ\gamma.$$

A special case of this relation is when $\{H, f\} = 0$: we then say that H and f are **in involution** and we find that $f(\gamma(t))$ is constant, i.e., is a first integral. This can be viewed as a version of Noether's theorem which relates a continuous group of symmetry to a conservation law. In this case the vector field ξ_f is the infinitesimal symmetry and '$f(\gamma(t)) =$ constant' is the conservation law.

2. The Lie bracket of two vector fields ξ_f and ξ_g is again a Hamiltonian vector field, more precisely

$$[\xi_f, \xi_g] = \xi_{\{f,g\}}.$$

This has the consequence that again if f and g are in involution, i.e., $\{f, g\} = 0$, then the flows of ξ_f and ξ_g commute.

Both properties together implies the following: assume that $\{f, H\} = 0$ and that (at least locally) df does vanish, which is equivalent to the fact that ξ_f does not vanish. Then we can reduce the number of variable by 2. A first reduction is due

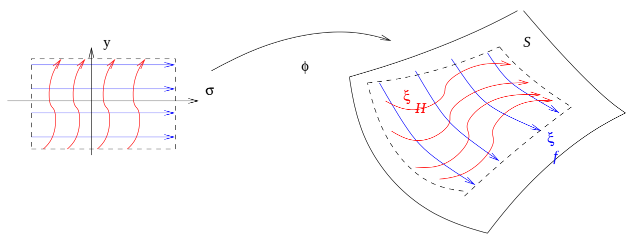

FIGURE 1. The symplectic reduction

to the first remark: the conservation of f along the integral curves of ξ_H can just be reformulated by saying that each integral curve of ξ_H is contained in a level set of f, i.e., the hypersurface $\mathcal{S} = \{m \in \mathbb{R}^{2n} |\ f(m) = C\}$. But also $\mathcal{S}$ is foliated by integral curves of the flow of ξ_f (a consequence of $\{f, f\} = 0$). So for any point $m_0 \in \mathcal{S}$ by the flow box theorem we can find a neighborhood $\mathcal{S}_{m_0}$ of m_0 in $\mathcal{S}$ and a diffeomorphism

$$\begin{aligned}\varphi:\quad (-\varepsilon,\varepsilon)\times B^{2n-2}(0,r) &\longrightarrow \mathcal{S}_{m_0}\\ (\sigma, y) &\longmapsto m\end{aligned}$$