ANALYTIC MANIFOLDS OF NONPOSITIVE CURVATURE

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Abstract. In this article we construct compact, real analytic Riemannian manifolds of nonpositive sectional curvature which have geometric rank one, but which contain a rich structure of totally geodesic subspaces of higher rank. Topologically the manifolds are obtained by blowing up certain, pairwise intersecting, codimension 2 submanifolds of a hyperbolic manifold. The metric on this blow–up is constructed explicitly by means of some Poincaré series, and appropriate methods for controlling its curvature and its rank are developed.

Résumé. Dans cet article sont construites des variétés riemanniennes analytiques compactes à courbure sectionnelle non-positive de rang géométrique un ayant une structure riche de sous-variétés totalement géodésiques de rangs plus élevés. Topologiquement ces variétés sont obtenues en éclatant certaines sous-variétés de codimension 2 d'une variété hyperbolique se coupant deux à deux. La métrique sur cet espace éclaté est construite explicitement grâce à des séries de Poincaré et des méthodes appropriées pour contrôler sa courbure et son rang sont développées.

M.S.C. Subject Classification Index (1991) : 53C20, 53C21.
GADGET

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1. INTRODUCTION

In this paper we construct new examples of compact real analytic Riemannian manifolds of nonpositive sectional curvature. The main result is

1.1. Theorem. — *Let $\mathbb{H}^n/\Gamma'$ be a compact manifold with constant curvature $K \equiv -1$ and let $\bar{\varrho}_i \in \mathrm{Iso}(\mathbb{H}^n/\Gamma')$, $1 \le i \le N$, be a family of rotations with fixed point sets*

$$\bar{V}_i := \mathrm{Fix}(\bar{\varrho}_i) = \{p \in \mathbb{H}^n/\Gamma' \mid \bar{\varrho}_i(p) = p\}$$

of codimension 2. Suppose that each $\bar{\varrho}_i$ permutes[1] *the N fixed point sets $\bar{V}_{i'}$. Moreover, for any pair of distinct fixed point sets $\bar{V}_{i_1}$ and $\bar{V}_{i_2}$ with $\bar{V}_{i_1} \cap \bar{V}_{i_2} \neq \emptyset$, it is required that $\bar{V}_{i_1} \cap \bar{V}_{i_2}$ has codimension 4 and that the intersection is orthogonal. Let $\pi\colon M \to \mathbb{H}^n/\Gamma'$ be the manifold obtained by blowing up $\bigcup_i \bar{V}_i$.*
Then, M carries a real analytic Riemannian metric g with sectional curvature $K \le 0$ everywhere and with $K < 0$ on the complement of $\pi^{-1}\big(\bigcup_{i=1}^N \bar{V}_i\big)$. The preimages $\hat{V}_i := \pi^{-1}(\bar{V}_i)$ and all their intersections $\hat{V}_I := \bigcap_{i\in I} \hat{V}_i$, $I \subset \{1,\dots,N\}$ are totally geodesic submanifolds of (M,g). Each projection $\pi_{(I)} := \pi_{|\hat{V}_I}$ factors through a Riemannian submersion $\hat{\pi}_{(I)}\colon \hat{V}_I \to \bar{V}_I^\star$ onto a space $\bar{V}_I^\star$ of nonpositive curvature. This submersion is a flat bundle over $\bar{V}_I^\star$ with totally geodesic fibres which are isometric to $\# I$–fold products of $\mathbb{RP}^1$'s of equal lengths.

The metric g will be constructed explicitly by means of a Poincaré series. For any subset $I \subset \{1,\dots N\}$ the holonomy of the flat bundle $\hat{\pi}_{(I)}\colon \hat{V}_I \to \bar{V}_I^*$ is determined by the holonomy of the normal bundle of $\bar{V}_I := \bigcap_{i\in I} \bar{V}_i \subset \mathbb{H}^n/\Gamma'$. Moreover, the existence of a single nonempty, totally geodesic submanifold $\hat{V}_i \subset M$ implies that the

[1] W.l.o.g. we may assume that each $\bar{\varrho}_i$ generates the maximal cyclic subgroup in $\mathrm{Iso}_0(\mathbb{H}^n/\Gamma')$ fixing $\bar{V}_i$. With this normalisation it is equivalent to require that the family $(\langle\bar{\varrho}_i\rangle)_{i=1}^N$ is closed under conjugation : for any pair (i_1,i_2) there exists i_3 such that $\bar{\varrho}_{i_1}\langle\bar{\varrho}_{i_2}\rangle\bar{\varrho}_{i_1}^{-1} = \langle\bar{\varrho}_{i_3}\rangle$.

fundamental group $\pi_1(M)$ is not hyperbolic in the sense of [GhH] and [Gr2]. However $\operatorname{rank}(\pi_1(M)) = 1$, where the rank of a finitely generated group Γ is defined in terms of the word metric d_Γ as follows (see [BE])

$$\operatorname{rank}(\Gamma) \geq k \;:\Leftrightarrow\; \exists C > 0 \;\forall \gamma \in \Gamma \;\exists \text{ a subgroup } A_\gamma \simeq \mathbb{Z}^k \text{with } d_\Gamma(\gamma, A_\gamma) \leq C.$$

Looking at the precise estimates for the curvature in Theorem 5.9 one can see that the metric g has *as little zero curvature as permitted by the fundamental group.* We shall explain this in more detail in Section 7.

To show that the hypotheses of Theorem 1.1 are not void, we quote from [AbSch] :

1.2. Theorem. — *Let Γ' be a torsion–free, normal subgroup of finite index in some cocompact, discrete group $\Gamma \subset \operatorname{Iso}(\mathbb{H}^n) = O^+(n,1)$. Suppose in addition that Γ contains commuting isometries $\varrho_1, \dots, \varrho_k$, whose fixed point sets are hyperbolic subspaces of codimension 2. If at most one of the ϱ_i's has order 2, then the induced rotations $\bar\varrho_i$ on $\mathbb{H}^n/\Gamma'$ satisfy the hypotheses of Theorem 1.1.*

In particular, there are *concrete examples*[2] of such groups $\Gamma' < \Gamma < \operatorname{Iso}(\mathbb{H}^n)$ and of rotations $\varrho_1, \dots, \varrho_k$ of this type with $n = 2k$. In this case the flat $\hat V_1 \cap \ldots \cap \hat V_k \subset M$ has the *maximal possible dimension* in view of the following general result proved at the end of Section 2.

1.3. Theorem. — *Let X^n be a simply–connected, real analytic Riemannian manifold with $K \leq 0$, and let $F^k \subset X^n$ be a k–flat of maximal dimension. Moreover, let $\Sigma_1, \dots, \Sigma_m \subset F^k$ be different singular hyperplanes through a common point p, where singular means that the set P_{Σ_i} of parallels to Σ_i is not contained in the flat F^k. Then,*

$$k + \sum_{i=1}^{m} \bigl(\dim P_{\Sigma_i} - k\bigr) \;\leq\; n \;. \tag{1.1}$$

Since $\dim P_{\Sigma_i} > k$, the number m of singular hyperplanes is estimated by the codimension $n-k$ of the flat F^k. In our example the strata $\hat V_{i_1} \cap \ldots \cap \hat V_{i_{k-1}}$ are

[2] In [Buy] S. Buyalo has used a slightly different approach to construct an interesting configuration of compact, codimension-2 subspaces $\bar V_i^2$ in the hyperbolic 120–cell space $\mathbb{H}^4/\Gamma'$ with the intersection pattern required for Theorem 1.1.

parallel sets of k different singular hyperplanes $\Sigma_{i_k} \subset \hat{V}_1 \cap \ldots \cap \hat{V}_k$. Thus Inequality (1.1) is sharp in this example.

The Weyl chamber structure of the flat $\hat{V}_1 \cap \ldots \cap \hat{V}_k$ is the same as the structure of a flat in the k–fold product $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$. An interesting open question is whether there are also real analytic manifolds of rank 1 with a maximal flat which has the Weyl chamber structure of the flat in an irreducible symmetric space.

We emphasize that the *crucial point* in Theorem 1.1 is the existence of a *real analytic* metric of nonpositive curvature on M. Indeed, it is much easier to obtain a C^∞–metric with $K \leq 0$ on M even *without assuming* that the codimension–2 submanifolds are fixed point sets of isometries. For completeness we state

1.4. Theorem. — *Let* $\left(\bar{V}_i\right)_{i=1}^N$ *be a finite family of compact, totally geodesically immersed submanifolds of codimension* 2 *in some compact hyperbolic space* $\mathbb{H}^n/\Gamma'$. *Suppose that the various sheets of* $\bigcup_i \bar{V}_i$ *intersect pairwise orthogonally in sets of codimension* 4, *if they intersect at all. Then, the blow–up* $\pi\colon M^n \to \mathbb{H}^n/\Gamma'$ *of* $\bigcup_i \bar{V}_i$ *carries a smooth metric with sectional curvature* $K \leq 0$.

The proof of Theorem 1.1 occupies Sections 3–6. The metric g in question is constructed explicitly in Theorem 3.7 and the relevant curvature estimates are the subject of Theorem 5.9.

The proof of Theorem 1.4 is much simpler, since all constructions can be done just locally. One could even give an independent proof based on a multiple warped product structure in the sense of [ONl, p. 210, Theorem 42].