THE PROBLEM OF GEODESICS, INTRINSIC DERIVATION AND THE USE OF CONTROL THEORY IN SINGULAR SUB-RIEMANNIAN GEOMETRY

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Abstract. We try to convince geometers that it is worth using Control Theory in the framework of sub-Riemannian structures, not only to get necessary conditions for length-minimizing curves, but also, from the very beginning, to give a description of sub-Riemannian structures by means of a global control vector bundle. This method is particularly efficient in characterizing admissible metrics with rank singularities. Some examples are developed.

Résumé. Notre but est d'essayer de convaincre les géomètres que cela vaut la peine d'appliquer les méthodes de la Théorie du Contrôle dans le contexte de structures sous-riemanniennes, non seulement pour obtenir des conditions nécessaires concernant les courbes minimisant la longueur, mais aussi, dès l'origine de la théorie, afin de définir globalement les structures sous-riemanniennes par des fibrés vectoriels dits de contrôle. Cette méthode est particulièrement efficace dans la caractérisation des métriques admissibles présentant des singularités de rang ; nous donnons des exemples.

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INTRODUCTION

1. Description of main results.

The main motivation for my talking here is to convince geometers that the Control Theory framework is providing a better understanding and an adapted tool in sub-Riemannian geometry. Our original presentation permits to associate to a singular plane distribution a family of natural sub-Riemannian metrics, with respect to which the regular case results are extendible to the singular one (section 4).

Another motivation is to give a really intrinsic definition in this context of a sub-Riemannian derivation (generalization of [S], section 8).

And the last motivation is to give an alternate proof that the abnormal horizontal helix in the Montgomery-Kupka example is length minimizing (section 9, [V], [V-P]). This method allows, as we know now, a generalization to any sub-Riemannian metric on a "generic" two distribution in $\mathbb{R}^3$.

Though looking far from the main concerns of Marcel Berger, the subject of this lecture has something to do with what has been a good deal of his own work ; namely, one of his successes has been the interpretation, in terms of Riemannian geometric invariants, of the asymptotic development of the heat kernel of the Laplace operator. In a parallel direction, G. Ben Arous [B-A], R. Léandre [L], G. Besson, (see also [A], [Bi], [G]) working on the asymptotic expansion of the Green kernel in the theory of hypo-elliptic operators, have pointed out the essential link between this expansion and the distance and geodesic notions in an associated regular or non regular plane distribution endowed with a Carnot-Carathéodory metric. The alternate name for such a framework is "sub-Riemannian geometry".

Anyway, geometers should be interested in sub-Riemannian structures for themselves, as did R.W. Brockett, R.S. Strichartz, C. Bär, U. Hamenstädt and also M. Gromov, P. Pansu, J. Mitchell, because they are nice particular examples of non integrable distributions on manifolds, besides the expansion of the Green kernel of hypo-elliptic operators.

One way of describing a regular or singular sub-Riemannian manifold M is providing M with a locally free, finite, constant rank p, bracket generating submodule $\mathcal{E}$ of the module of vector fields $\chi(M)$. An absolutely continuous (a.c.) curve is called **horizontal** if its velocity vector lies a.e. in $\mathcal{E}$.

Chow's theorem [C], using the bracket generating condition, says that the space of horizontal piecewise C^1-curves joining two fixed points x_0 and x_1 is not empty.

The two main problems are then,

(i) among the a.c. horizontal curves joining x_0 and x_1, does there exist some length minimizing curve ?

(ii) if yes, how to characterize these curves ?

Now, provided the Riemannian manifold $(M, \mathbf{g})$ is complete, it is well-known, in the regular case, that the minimum exists and that standard variational methods of Riemannian Geometry do not solve the sub-Riemannian minimization problem. In contrast to the Riemannian case, where the energy minimizing curves are characterized as solution of a differential system (**G**), here, both notions can be generalized but they are no longer equivalent [S]. The Maximum Principle of Control Theory was already known as a very good tool giving account of "abnormal" geodesics, i.e., curves minimizing the energy between two given points but not verifying the differential "geodesic" equation (**G**), generalizing the Riemannian geodesic equation obtained by a classical variational principle (this was already realized in the regular case, see [Br], [S], see also [Gr], [Mi]).

Here, we are using Control Theory from the very beginning of the definition of **singular,** i.e., not constant rank, plane-distribution. This last setting out is original and allows plenty of sub-Riemannian metrics on a given plane distribution. The main result is showing the link between metric and distribution in the neighbourhood of singularities through the Control space ; in the regular case, any sub-Riemannian metric can be seen as the restriction to the plane distribution of some (actually infinetely many) Riemannian metrics on M, whereas **in the singular case, given any sub-Riemannian metric, there exists no Riemannian metric on M, such that its restriction to the plane distribution could be the given one**.

In section 2, we give an account of what is known about regular sub-Riemannian manifolds M (the plane distribution is then of constant rank).

In section 3, we do our best to give a quick survey of the main ideas explaining how the maximum principle works, following the inventors of the theory, see [P].

In section 4, we use, from the beginning, ideas of Optimal Control Theory and describe the framework of a singular sub-Riemannian geometry, where the "horizontal" singular distribution is generated by a module of vector fields, locally free of finite rank p (definition (4-6)) ; possible metrics on such a plane distribution have to be chosen carefully, otherwise **the distance between two given distinct points of the singular set in M could be zero or never be achieved by any horizontal curve**, as illustrated by means of the very simple Example (4-1).

In section 5, we merely prove that, even in this context, looking for a horizontal length minimizing curve among horizontal a.c. curves $\gamma : I \longrightarrow M$ joining two fixed points x_0 and x_1, is equivalent to looking for a horizontal energy minimizing curve between x_0 and x_1. The first one is defined up to a.c. reparametrizations. One of these provides the curve with a velocity vector of constant norm and is then energy minimizing.

In section 6, we prove, applying Bellaïche's method to this context [Bel], that between two distinct points, within a compact cell K, the minimum of energy is finite and is actually achieved on some curve.

In section 7, we use the Maximum Principle, knowing that the minimum of energy is achieved on some curve to display necessary conditions in the form of differential equations or conditions involving derivatives which are to be defined carefully in this case. The result is that there exist three kinds of minimizing curves, either normal (N) or strictly abnormal (SAN), or both (NAN), exactly as in regular sub-Riemannian geometry. Conversely, a curve satisfying the (N) or (NAN) condition are locally energy minimizing curves, but as far as we know, there does not exist criteria to tell when a (SAN)-curve is locally length minimizing or not. Actually, we have now (1993) examples of a non length-minimizing (SAN)-curve for some codimension one distributions in $\mathbb{R}^{2p}$ (see [P-V-2]). Since the end of 1993 we know also that, in dimension 3, the Montgomery example is a generic local model : the abnormal horizontal curves drawn on the singular surface are (NAN) or (SAN), always C^1-rigid, and locally minimizing, whatever the sub-Riemannian metric [V-P]. Finally,