CUT LOCI AND DISTANCE SPHERES ON ALEXANDROV SURFACES

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Abstract. The purpose of the present paper is to investigate the structure of distance spheres and cut locus $C(K)$ to a compact set K of a complete Alexandrov surface X with curvature bounded below. The structure of distance spheres around K is almost the same as that of the smooth case. However $C(K)$ carries different structure from the smooth case. As is seen in examples of Alexandrov surfaces, it is proved that the set of all end points $C_e(K)$ of $C(K)$ is not necessarily countable and may possibly be a fractal set and have an infinite length. It is proved that all the critical values of the distance function to K is closed and of Lebesgue measure zero. This is obtained by proving a generalized Sard theorem for one-valuable continuous functions.

Our method applies to the cut locus to a point at infinity of a noncompact X and to Busemann functions on it. Here the structure of all co-points of asymptotic rays in the sense of Busemann is investigated. This has not been studied in the smooth case.

Résumé. L'objet de cet article est d'étudier la structure des sphères de distance et du cut locus $C(K)$ d'un ensemble compact.

M.S.C. Subject Classification Index (1991) : 53C20.

Acknowledgements. Research of the authors was partially supported by Grant-in-Aid for Co-operative Research, Grant No. 05302004

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INTRODUCTION

The topological structure of the cut locus $C(p)$ to a point p on a complete, simply connected and real analytic Riemannian 2-manifold M was first investigated by Poincaré [P], Myers [M1], [M2] and Whitehead [W]. If such an M has positive Gaussian curvature, then (1) Poincaré proved that $C(p)$ is a union of arcs and does not contain any closed curve and its endpoints are at most finite which are conjugate to p, and (2) Myers proved that if M is compact (and hence homeomorphic to a 2-sphere), then $C(p)$ is a tree and if M is noncompact (and hence homeomorphic to $\mathbb{R}^2$), then it is a union of trees. Here, a topological set T is by definition a *tree* iff any two points on T is joined by a unique Jordan arc in T. A point x on a tree T is by definition an *endpoint* iff $T \setminus \{x\}$ is connected. Whitehead proved that if M is not simply connected, then $C(p)$ carries the structure of a local tree and the number of cycles in $C(p)$ coincides with the first Betti number of M. Here, a topological set C is by definition a *local tree* iff for every point $x \in C$ and for every neighborhood U around x, there exists a smaller neighborhood $T \subset U$ around x which is a tree.

The structure of geodesic parallel circles for a simple closed curve $\mathcal{C}$ in a real analytic Riemannian plane M was first investigated by Fiala [F] in connection with an isoperimetric inequality. Hartman extended Fiala's results (and also Myers' ones on $C(p)$) to a Riemannian plane with C^2-metric. Geodesic parallel coordinates for a given simply closed C^2-curve was employed in [H] to prove that there exists a closed set $\mathcal{E} \subset [0, \infty)$ of measure zero such that if $t \notin \mathcal{E}$, then

(1) the geodesic t-sphere $\mathcal{S}(\mathcal{C}; t) := \{\, x \in M \,;\, d(x, \mathcal{C}) = t\}$ around $\mathcal{C}$ consists of a finite disjoint union of piecewise C^2-curves each component of which is homeomorphic to a circle,

(2) the length $L(t)$ of $\mathcal{S}(\mathcal{C}; t)$ exists, and moreover $\frac{dL(t)}{dt}$ also exists and is continuous on $(0, \infty) \setminus \mathcal{E}$. Furthermore, the set $\mathcal{E}$ is determined by the topological structure of the cut locus and focal locus to $\mathcal{C}$.

These results were extended to complete, open and smooth Riemannian 2-manifolds (finitely connected or infinitely connected) in [S], [ST1], [ST2].

The purpose of the present article is to establish almost similar results on the structure of cut loci and geodesic spheres without assuming almost any differentiability. In fact, a simple closed curve in a C^2-Riemannian plane will be replaced in our results by a compact set in an Alexandrov surface. *From now on, let X be a connected and complete Alexandrov space without boundary of dimension 2 whose curvature is bounded below by a constant k.* Let $K \subset X$ be an arbitrary fixed compact set and $\rho : X \to \mathbb{R}$ the distance function to K. Let $\mathcal{S}(t) := \rho^{-1}(t)$ for $t > 0$ be the distance t-sphere of K. Let $C(K)$ be the cut locus to K and $C_e(K)$ the set of all endpoints of $C(K)$. With these notations our results are stated as follows.

Theorem A. — *For a connected component $C_0(K)$ of $C(K)$,*

(1) *$C_0(K)$ carries the structure of a local tree and any two points on it can be joined by a rectifiable Jordan arc in it ;*

(2) *the inner metric topology of $C_0(K)$ is equivalent to the induced topology from X ;*

(3) *there exists a class $\mathcal{M} := \{m_1, \cdots\}$ of countably many rectifiable Jordan arcs $m_i : I_i \to C_0(K)$, $i = 1, \cdots$, such that I_i is an open or closed interval and such that*

$$C_0(K) \setminus C_e(K) = \bigcup_{i=1}^{\infty} m_i(I_i) \ , \quad \textit{disjoint union} \ ;$$

(4) *each m_i has at most countably many branch points such that there are at most countably many members in $\mathcal{M}$ emanating from each of them.*

The above result is *optimal* in the sense that $C(K)$ in Example 4 cannot be covered by any countable union of Jordan arcs.

We see from (3) and (4) in Theorem A that $C(K)$ has, roughly speaking, a self similarity. The cut locus $C_0(K)$ is a *fractal set* iff the Hausdorff dimension of $C_0(K)$ in X is not an integer. Example 4 in §1 suggests that $C_0(K)$ will be a fractal set, where $C_e(K)$ is uncountable.

Theorem B. — *There exists a set $\mathcal{E} \subset (0, \infty)$ of measure zero with the following properties. For every $t \notin \mathcal{E}$,*

(1) *$\mathcal{S}(t)$ consists of a disjoint union of finitely many simply closed curves.*

(2) *$\mathcal{S}(t)$ is rectifiable.*

(3) *Every point $x \in \mathcal{S}(t) \cap C(K)$ is joined to K by at most two distinct geodesics of the same length t. Furthermore, if $x \in C(K) \cap \mathcal{S}(t)$ is joined to K by a unique geodesic, then $x \in C_e(K)$.*

(4) *There exists at most countably many points in $\mathcal{S}(t) \cap C(K)$ which are joined to K by two distinct geodesics.*

It should be noted that in contrast with the Riemannian case, the set $\mathcal{E}$ is not always closed. In fact, X admits a singular set $\mathrm{Sing}(X)$ and $\mathcal{E}$ contains $\rho(\mathrm{Sing(X)})$. Example 2 in §1 provides the case where $\rho(\mathrm{Sing}(X))$ is a dense set on $(0, \mathrm{diam}\, X)$.

In due course of the proof we obtain a generalized Sard theorem on the set of all critical values of a continuous (not necessarily of bounded variation) function, see Lemma 3.2, and prove the

Theorem C. — *The set of all critical values of the distance function to K is closed and of measure zero.*

The Basic Lemma applies to the cut locus of a point at infinity. Let $\gamma : [0, \infty) \to X$ be an arbitrary fixed ray. A *co-ray σ to γ* is by definition a ray obtained by the limit of a sequence of minimizing geodesics $\sigma_j : [0, \ell_j] \to X$ such that $\lim_{j\to\infty} \sigma_j(0) = \sigma(0)$ and such that $\{\sigma_j(\ell_j)\}$ is a monotone divergent sequence on $\gamma[0, \infty)$. Through every point on X there passes at least a co-ray to γ. A co-ray σ to γ is said to be *maximal* iff it is not properly contained in any co-ray to γ. Let $C(\gamma(\infty))$ be the set of all the starting points of all maximal co-rays to γ. In the Riemannian case the set $C(\gamma(\infty))$ is contained in the set of all non-differentiable points of the Busemann function F_γ with respect to γ. Here F_γ is defined by

$$F_\gamma(x) := \lim_{t\to\infty} [t - d(x, \gamma(t))], \qquad x \in X \ .$$

The set $C(\gamma(\infty))$ may be understood as the cut locus at a point $\gamma(\infty)$ of infinity, for it carries the same structure as cut locus. The structure of $C(\gamma(\infty))$ has not been discussed even in Riemannian case. Our proof method applies to investigate the structure of $C(\gamma(\infty))$ on X, and we obtain