DISCRETIZATION OF POSITIVE HARMONIC FUNCTIONS ON RIEMANNIAN MANIFOLDS AND MARTIN BOUNDARY

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Abstract. Let X be a separated subset in a connected Riemannian manifold M with bounded geometry such that the ε-neighbourhood of X is recurrent w.r.t. Brownian motion on M for some $\varepsilon > 0$. The main result of this paper says that the data in the discretization procedure of Lyons and Sullivan can be chosen such that the Green function of M and the resulting Markov chain on X coincide up to a constant on pairs (y, z), where $y \neq z$ are points in X.

Résumé. Soit X un sous-ensemble séparé d'une variété riemannienne M à géométrie bornée tel que le voisinage d'épaisseur ε de X est récurrent pour le mouvement brownien sur M pour au moins un ε positif. Le principal résultat de cet article dit que les données du procédé des discrétisations de Lyons et Sullivan peuvent être choisies de telle sorte que la fonction de Green de M et la chaîne de Markov sur X qui s'en déduit coïncident à une constante près sur les paires de points (y, z) avec $y \neq z$.

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INTRODUCTION

We are interested in the connection between potential theory of the Laplacian on Riemannian manifolds and the potential theory of Markov chains on discrete subsets. Such a connection has been established by Furstenberg [F] in the case of discrete subgroups of $Sl(2,\mathbb{R})$. We investigate the discretization procedure of Lyons and Sullivan [LS], which associates to a so-called $*$-*recurrent* (respectively *cocompact*) discrete subset X of a connected Riemannian manifold M a family of probability measures $\mu_y,\ y \in M$, on X such that

$$H(y) = \mu_y(H) := \sum_{x\in X} H(x)\mu_y(x)$$

for any bounded (respectively positive) harmonic function H on M. In particular, the restriction of H to X is a μ-harmonic function with respect to the Markov chain on X defined by the measures $\mu_x, \quad x \in X$ (that is, $\mu_x(H) = H(x)$ for all $x \in X$). Under some extra assumptions on the data involved in the construction, one obtains in this way all bounded (respectively positive) μ-harmonic functions on X (see [A], [K]) and, if X is cocompact, that Brownian motion on M is transient iff the Markov chain on X is transient [LS].

A more precise information about behaviour at infinity of harmonic functions is given by the Martin compactification $cl_\Delta M$ and the Martin boundary $\partial_\Delta M$ of M. By definition, $cl_\Delta M = M\dot\cup\partial_\Delta M$ is the closure of M in the space of positive superharmonic functions via the embedding $y \longmapsto K(.,y)$, where

$$K(.,y) = G(.,y)/G(x_0,y)$$

is the Martin kernel, G is the Green function of M and $x_0 \in M$ is a chosen origin. For convenience, we choose $x_0 \in X$. The Martin compactification $cl_\mu X$ and Martin boundary $\partial_\mu X$ of X with respect to a Markov chain on X are defined in the same way by using the Martin kernel k and the Green function g of the Markov chain. The definition of the Martin boundary requires that Brownian motion on M (respectively the Markov chain on X) has a Green function, i.e., that it is transient.

As a consequence of Theorems 1.11, 2.7, 2.8, 3.1 and Corollary 2.9 below we obtain the theorem

Main theorem. — *Assume that the geometry of M is bounded and that X is a discrete subset of M such that, for some $\varepsilon > 0$,*

(i) $dist(x, z) \geq 2\varepsilon$ for all $x \neq z$ in X; (ii) $B_\varepsilon(X)$ is recurrent.

Then, for some appropriate choice of data, the measures $\mu_y, y \in M$, of Lyons and Sullivan satisfy

(a) for some positive constant κ we have $g(x, z) = \kappa G(x, z)$ for all $x \neq z$ in X. In particular, the Markov chain on X is transient iff Brownian motion on M is.

If Brownian motion on M is transient, then $\mu_x(z) = \mu_z(x)$ for all x, z in X and

(b) the inclusion $X \subset M$ extends to a homeomorphism of $cl_\mu X$ and $\overline{X}$, where $\overline{X}$ is the closure of X in $cl_\Delta M$;

(c) restriction defines an isomorphism between the simplex of positive harmonic functions on M spanned by $\overline{X} \cap \partial_\Delta M$ and the space of positive μ-harmonic functions on X which are 1 at x_0.

The Harnack inequality implies that $\overline{X} \cap \partial_\Delta M$ contains all extremal positive harmonic functions of M which are 1 at x_0 if X is a net, that is, if $B_R(X) = M$ for some $R > 0$. Thus (c) implies in this case that the space of positive harmonic functions on M and the space of positive μ-harmonic functions on X are isomorphic, a result due to Ancona [A].

If Γ is a discrete group of isometries of M and X is the orbit of a point x_0 on which Γ acts freely, then X satisfies (i). Property (ii) holds if $vol(M/\Gamma) < \infty$ or , more generally, if the Brownian motion on M/Γ is recurrent. If this is the case, then the Markov chain on X corresponds to a (left-invariant) symmetric random walk on Γ (via the natural identification of Γ and $X = \Gamma(x_0)$).

Corollary. — *There exists a symmetric random walk on the free group F_q with $q \geq 2$ generators with Martin boundary equal to a circle.*

As for the proof, recall that the Martin boundary of the hyperbolic plane H^2 is the circle (at infinity) and that F_q acts as a discrete group of isometries on H^2 with $vol(H^2/F_q) < \infty$.

It follows from Theorem 3.2 below that the measure defining the random walk on F_q has finite logarithmic moment with respect to the word norm on F_q and finite entropy. This has to be contrasted with the case of probabilities on F_q with finite support, for which the Martin boundary is known to be a Cantor set [D].

We would like to thank Martine Babillot to whom we owe the assertion and the proof of the symmetry of the measures μ_x in the above theorem. The second author gratefully acknowledges the support by the SFB 256 at the University of Bonn.

1. HARMONIC FUNCTIONS

Let M denote a connected Riemannian manifold. A *Brownian path* on M is a continuous curve

$$\omega : [0, \zeta(\omega)) \to M, \text{ where } \zeta(\omega) \in (0, \infty] \ .$$

For x in M, let P_x denote the measure on the space of Brownian paths on M with $\omega(0) = x$ defining the Brownian motion on M starting from x. For a Borel measure λ on M let P_λ be defined by $P_\lambda = \int P_x \lambda(dx)$. The measure P_λ describes the Brownian motion on M with initial distribution λ.

For a closed subset F of M and a Brownian path ω set

$$R^F(\omega) = inf\{t \geq 0 \ \mid \ \omega(t) \in F\} \ .$$

The *balayage* $\beta_\lambda^F = \beta(\lambda, F)$ of a measure λ onto F is the distribution of P_λ at the time R^F, i.e., for A a Borel subset of M,

$$\beta_\lambda^F(A) = P_\lambda\{\omega \ \mid \ R^F(\omega) < \zeta(\omega) \text{ and } \omega(R^F(\omega)) \in A\} \ .$$

For short we set $\beta_x^F = \beta(x, F) = \beta(\delta_x, F)$; then $\beta(\lambda, F) = \int \beta(x, F)\lambda(dx)$. For x in F, we have $\beta(x, F) = \delta_x$. We say that F is *recurrent* if $\beta_x^F(F) = 1$ for all x in M.

For an open subset V of M and a Brownian path ω set

$$S^V(\omega) = inf\{t \geq 0 \ \mid \ \omega(t) \in M \backslash V\} \ .$$