Séminaires & Congrès
4, 2000, p. 35–52

TWISTOR AND KILLING SPINORS IN LORENTZIAN GEOMETRY

by

Helga Baum

Abstract. — This paper is a survey of recent results concerning twistor and Killing spinors on Lorentzian manifolds based on lectures given at CIRM, Luminy, in June 1999, and at ESI, Wien, in October 1999. After some basic facts about twistor spinors we explain a relation between Lorentzian twistor spinors with lightlike Dirac current and the Fefferman spaces of strictly pseudoconvex spin manifolds which appear in CR-geometry. Secondly, we discuss the relation between twistor spinors with timelike Dirac current and Lorentzian Einstein Sasaki structures. Then, we indicate the local structure of all Lorentzian manifolds carrying real Killing spinors. In particular, we show a global Splitting Theorem for complete Lorentzian manifolds in the presence of Killing spinors. Finally, we review some facts about parallel spinors in Lorentzian geometry.

***Résumé* (Twisteurs et spineurs de Killing en géométrie lorentzienne).** — Le présent papier est un article de synthèse basé sur les exposés donnés au CIRM, Luminy, en juin 1999, et à l'ESI, Vienne, en octobre 1999, concernant des nouveaux résultats sur les spineurs twisteurs et les spineurs de Killing lorentziens. Après quelques préliminaires sur les spineurs twisteurs, on met en évidence des relations entre les spineurs twisteurs lorentziens admettant un courant de Dirac isotrope et les espaces de Fefferman des variétés spinorielles strictement pseudoconvexes qui apparaissent dans la géométrie CR. De plus, on décrit la relation entre les spineurs twisteurs admettant un courant de Dirac de type temps et les structures de Sasaki-Einstein lorentziennes. On indique aussi la structure locale des variétés lorentziennes admettant des spineurs de Killing réels. En particulier, on obtient un théorème de « splitting » global pour les variétés lorentziennes complètes qui admettent des spineurs de Killing. Enfin, on fait le point sur la théorie des spineurs parallèles en géométrie lorentzienne.

2000 *Mathematics Subject Classification*. — 58G30, 53C50, 53A50.
Key words and phrases. — Twistor equation, twistor spinors, Killing spinors, parallel spinors, Lorentzian manifolds, CR-geometry, Fefferman spaces, Lorentzian Einstein-Sasaki manifolds, holonomy groups.

1. Introduction

Twistor spinors were introduced by R.Penrose and his collaborators in General Relativity as solutions of a conformally invariant spinorial field equation (twistor equation) (see [**Pen67**], [**PR86**], [**NW84**]). Twistor spinors are also of interest in physics since they define infinitesimal isometries in semi-Riemannian supergeometry (see [**ACDS98**]). In Riemannian geometry the twistor equation first appeared as an integrability condition for the canonical almost complex structure of the twistor space of an oriented four-dimensional Riemannian manifold (see [**AHS78**]). In the second half of the 80's A.Lichnerowicz started the systematic investigation of twistor spinors on Riemannian spin manifolds from the view point of conformal differential geometry. Nowadays one has a lot of structure results and examples for manifolds with twistor spinors in the Riemannian setting (see e.g. [**Lic88b**], [**Lic88a**], [**Lic89**], [**Wan89**], [**Fri89**] [**Lic90**], [**BFGK91**], [**Hab90**], [**Bär93**], [**Hab94**], [**Hab96**], [**KR94**], [**KR96**], [**KR97b**], [**KR97a**], [**KR98**]).
An other special kind of spinor fields related to Killing vector fields and Killing tensors and therefore called Killing spinors is used in supergravity and superstring theories (see e.g. [**HPSW72**], [**DNP86**], [**FO99a**], [**AFOHS98**]). In mathematics the name Killing spinor is used (more restrictive than in physics literature) for those twistor spinors which are simultaneous eigenspinors of the Dirac operator. The interest of mathematicians in Killing spinors started with the observation of Th. Friedrich in 1980 that a special kind of Killing spinors realise the limit case in the eigenvalue estimate of the Dirac operator on compact Riemannian spin manifolds of positive scalar curvature. In the time after the Riemannian geometries admitting Killing spinors were intensively studied. They are now basically known and in low dimensions completely classified (see [**BFGK91**] [**Hij86**], [**Bär93**]). These results found applications also outside the spin geometry, for example as tool for proving rigidity theorems for asymptotically hyperbolic Riemannian manifolds (see [**AD98**], [**Her98**]). In the last years the investigation of special adapted spinorial field equations was extended to Kähler, quaternionic-Kähler and Weyl geometry (see e.g. [**MS96**], [**Mor99**], [**KSW98**], [**Buc00b**], [**Buc00a**]).

In opposite to the situation in the Riemannian setting, there is not much known about solutions of the twistor and Killing equation in the *pseudo*-Riemannian setting, where these equations originally came from. The general indefinite case was studied by Ines Kath in [**Kat00**], [**Kat98**], [**Katb**], [**Kata**], where one can find construction principles and examples for indefinite manifolds carrying Killing and parallel spinors. In the present paper we restrict ourselves to the Lorentzian case. We explain some results concerning the twistor and Killing equation in *Lorentzian* geometry, which we obtained in a common project with Ines Kath, Christoph Bohle, Felipe Leitner and Thomas Leistner.

2. Basic facts on twistor spinors

Let $(M^{n,k}, g)$ be a smooth semi-Riemannian spin manifold of index k and dimension $n \geq 3$ with the spinor bundle S. There are two conformally covariant differential operators of first order acting on the spinor fields $\Gamma(S)$, the Dirac operator D and the twistor operator (also called Penrose operator) P. The Dirac operator is defined as the composition of the spinor derivative ∇^S with the Clifford multiplication μ

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{\mu} \Gamma(S),$$

whereas the twistor operator is the composition of the spinor derivative ∇^S with the projection p onto the kernel of the Clifford multiplication μ

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \overset{g}{\approx} \Gamma(TM \otimes S) \xrightarrow{p} \Gamma(\ker \mu).$$

The elements of the kernel of P are called *twistor spinors*. A spinor field φ is a twistor spinor if and only if it satisfies the *twistor equation*

$$\nabla^S_X \varphi + \frac{1}{n} X \cdot D\varphi = 0$$

for each vector field X. Special twistor spinors are the parallel and the Killing spinors, which satisfy simultaneous the Dirac equation. They are given by the spinorial field equation

$$\nabla^S_X \varphi = \lambda\, X \cdot \varphi\,, \quad \lambda \in \mathbb{C}.$$

The complex number λ is called Killing number.

We are interested in the following geometric problems:

1. *Which semi-Riemannian (in particular Lorentzian) geometries admit solutions of the twistor equation?*
2. *How the properties of twistor spinors are related to the geometric structures where they can occur.*

The basic property of the twistor equation is that it is conformally covariant: Let $\tilde{g} = e^{2\sigma} g$ be a conformally equivalent metric to g and let the spinor bundles of (M, g) and $(M, \tilde{g})$ be identified in the standard way. Then for the twistor operators of P and $\tilde{P}$ the relation

$$\tilde{P}\varphi = e^{-\frac{1}{2}\sigma} P(e^{-\frac{1}{2}\sigma}\varphi)$$

holds.

Let us denote by R the scalar curvature and by Ric the Ricci curvature of $(M^{n,k}, g)$. K denotes the Rho tensor

$$K = \frac{1}{n-2}\left\{\frac{R}{2(n-1)} g - \mathrm{Ric}\right\}.$$

We always identify TM with TM^* using the metric g. For a $(2,0)$-tensor field B we denote by the same symbol B the corresponding $(1,1)$-tensor field $B : TM \longrightarrow TM$,

$g(B(X),Y) = B(X,Y)$. Let C be the (2,1)-Cotton-York tensor

$$C(X,Y) = (\nabla_X K)(Y) - (\nabla_Y K)(X).$$

Furthermore, let W be the (4,0)-Weyl tensor of (M,g) and let denote by the same symbol the corresponding (2,2)-tensor field $W : \Lambda^2 M \longrightarrow \Lambda^2 M$. Then we have the following integrability conditions for twistor spinors

Proposition 2.1 **([BFGK91,** Th.1.3, Th.1.5]**).** — *Let $\varphi \in \Gamma(S)$ be a twistor spinor and $\eta = Y \wedge Z \in \Lambda^2 M$ a two form. Then*

$$D^2\varphi = \frac{1}{4}\frac{n}{n-1} R\varphi \tag{1}$$

$$\nabla^S_X D\varphi = \frac{n}{2} K(X) \cdot \varphi \tag{2}$$

$$W(\eta) \cdot \varphi = 0 \tag{3}$$

$$W(\eta) \cdot D\varphi = n\, C(Y,Z) \cdot \varphi \tag{4}$$

$$(\nabla_X W)(\eta) \cdot \varphi = X \cdot C(Y,Z) \cdot \varphi + \frac{2}{n}(X \lrcorner\ W(\eta)) \cdot D\varphi \tag{5}$$

If (M^n, g) admits Killing spinors the Ricci and the scalar curvature of M satisfy in addition

Proposition 2.2. — *Let $\varphi \in \Gamma(S)$ be a Killing spinor with the Killing number $\lambda \in \mathbb{C}$. Then*

1. $(\mathrm{Ric}(X) - 4\lambda^2(n-1)X) \cdot \varphi = 0$. *In particular, the image of the endomorphism* $\mathrm{Ric} - 4\lambda^2(n-1) id_{TM}$ *is totally lightlike.*
2. *The scalar curvature is constant and given by* $R = 4n(n-1)\lambda^2$. *The Killing number λ is real or purely imaginary.*

If the Killing number λ is zero ($R = 0$), φ is a parallel spinor, in case λ is real and non-zero ($R > 0$), φ is called real Killing spinor, and in case λ is purely imaginary ($R < 0$), φ is called imaginary Killing spinor.
We consider the following covariant derivative in the bundle $E = S \oplus S$

$$\nabla^E_X := \begin{pmatrix} \nabla^S_X & \frac{1}{n} X \cdot \\ -\frac{n}{2} K(X) & \nabla^S_X \end{pmatrix}.$$

Using the integrability condition (2) of Proposition 2.1 one obtains the following

Proposition 2.3 **([BFGK91,** Th.1.4]**).** — *For any twistor spinor φ it holds $\nabla^E \binom{\varphi}{D\varphi} = 0$. Conversely, if $\binom{\varphi}{\psi}$ is ∇^E-parallel, then φ is a twistor spinor and $\psi = D\varphi$.*

The calculation of the curvature of ∇^E and Proposition 2.3 yield

Proposition 2.4. — *The dimension of the space of twistor spinors is conformally invariant and bounded by*

$$\dim \ker P \leq 2^{[\frac{n}{2}]+1} = 2 \cdot \mathrm{rank}\, S =: d_n.$$

For each simply connected, conformally flat semi-Riemannian spin manifold the dimension of the space of twistor spinors equals d_n. On the other hand, the maximal dimension d_n can only occur if (M,g) is conformally flat.

Let $M^{n,k}$ be a conformally flat manifold with the universal covering $\tilde{M}^{n,k}$. The bundle E is a tractor bundle associated to the conformal structure of (M,g) and ∇^E is the covariant derivative on E defined by the normal conformal Cartan connection. (For the definition of tractor bundles see for example [**CG99**]). Using this description one obtains a development of $\tilde{M}^{n,k}$ into a covering $\hat{C}^{n,k}$ of the (pseudo-) Möbius sphere. The corresponding holonomy representation

$$\rho : \pi_1(M) \longrightarrow \mathrm{O}(k+1, n-k+1)$$

of the fundamental group of M characterizes conformally flat spin manifolds with twistor spinors.

***Proposition 2.5* ([KR97a], [Lei00b]).** — *A conformally flat semi-Riemannian manifold is spin and admits twistor spinors iff the holonomy representation ρ admits a lift*

$$\tilde{\rho} : \pi_1(M) \longrightarrow \mathrm{Spin}(k+1, n-k+1)$$

and the the representation of $\pi_1(M)$ on the spinor module $\Delta_{k+1,n-k+1}$ has a proper trivial subrepresentation.

If the scalar curvature R of $(M^{n,k}, g)$ is constant and non-zero, the integrability conditions (1) and (2) of Proposition 2.1 show that the spinor fields

$$\psi_{\pm} := \frac{1}{2}\varphi \pm \sqrt{\frac{n-1}{nR}}\, D\varphi$$

are formal eigenspinors of the Dirac operator D to the eigenvalue $\pm\frac{1}{2}\sqrt{\frac{nR}{n-1}}$.
For an Einstein space $(M^{n,k}, g)$ with constant scalar curvature $R \neq 0$ the spinor fields $\psi_{\pm}$ are Killing spinors to the Killing number $\lambda = \mp\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}$. Hence, on this class of semi-Riemannian manifolds each twistor spinor is the sum of two Killing spinors.

To each spinor field φ we associate a vector field V_{φ} (Dirac current) by the formula

$$g(V_{\varphi}, X) := i^{k+1}\langle X \cdot \varphi, \varphi\rangle\,, \qquad X \in \Gamma(TM).$$

***Proposition 2.6*.** — *Let $\varphi \in \Gamma(S)$ be a twistor spinor. Then V_{φ} is a conformal vector field with the divergence*

$$\mathrm{div}(V_{\varphi}) = -2(-1)^{[\frac{k}{2}]}\, h(\langle D\varphi, \varphi\rangle)\,,$$

where $h(f)$ denotes the real part of f if the index k of g is odd and the imaginary part of f, if the index k of g is even.