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PSEUDOVALUATIONS ON THE DE RHAM–WITT COMPLEX

by Rubén Muñoz--Bertrand

Abstract. — For a polynomial ring over a commutative ring of positive characteristic, we define on the associated de Rham–Witt complex a set of functions, and show that they are pseudovaluations in the sense of Davis, Langer and Zink. To achieve this, we explicitly compute products of basic elements on the complex. We also prove that the overconvergent de Rham–Witt complex can be recovered using these pseudovaluations.

Résumé (*Pseudovaluations sur le complexe de de Rham–Witt*). — Pour tout anneau polynomial sur un anneau commutatif de caractéristique strictement positive, on définit sur le complexe de de Rham–Witt associé un ensemble de fonctions, et l'on démontre que ce sont des pseudovaluations au sens de Davis, Langer et Zink. Pour y parvenir, on calcule explicitement des produits d'éléments basiques du complexe. On prouve également que le complexe de de Rham–Witt surconvergent peut être retrouvé en employant ces pseudovaluations.

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Introduction

Davis, Langer and Zink introduced the overconvergent de Rham–Witt complex in [2]. It is a complex of sheaves defined on any smooth variety X over a perfect field k of positive characteristic. It can be used to compute both the Monsky–Washnitzer and the rigid cohomology of the variety. This comparison was first established by [2] for quasi-projective smooth varieties; the assumption of quasi-projectiveness was then removed by Lawless [7].

This complex is defined as a differential graded algebra (dga) contained in the de Rham–Witt complex $W\Omega^\bullet_{X/k}$ of Deligne and Illusie. In order to achieve this they defined for any $\varepsilon > 0$, in the case where X is the spectrum of a polynomial ring $k[\underline{X}]$ over k, an order function $\gamma_\varepsilon\colon W\Omega^\bullet_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\}$. The overconvergent de Rham–Witt complex of X is the set of all $x \in W\Omega^\bullet_{k[\underline{X}]/k}$, such that $\gamma_\varepsilon(x) \neq -\infty$, for some $\varepsilon > 0$. In the general case, it is defined as the functional image of this set for a surjective morphism of smooth commutative algebras over k.

In degree zero (that is, for Witt vectors), these maps have nice properties. One of these is that they are pseudovaluations. We recall the definition. A **pseudovaluation** on a ring R is a function $v\colon R \to \mathbb{R} \cup \{+\infty, -\infty\}$, such that:

$$v(0) = +\infty,$$
$$v(1) = 0,$$
$$\forall r \in R,\ v(r) = v(-r)\,,$$
$$\forall r, s \in R,\ v(r+s) \geqslant \min\{v(r)\,, v(s)\}\,,$$
$$\forall r, s \in R,\ (v(r) \neq -\infty) \wedge (v(s) \neq -\infty) \implies (v(rs) \geqslant v(r) + v(s))\,.$$

The last formula will be referred to as the product formula in the remainder of this article.

Pseudovaluations and their behaviour have been studied in [3]. It appears that they form a convenient framework to study the overconvergence of recursive sequences. However, there are counterexamples showing that in positive degree, the maps γ_ε are not pseudovaluations. This becomes an obstacle when one wants to study the local structure of the overconvergent de Rham–Witt complex, or when one tries to find an interpretation of F-isocrystals for the overconvergent de Rham–Witt complex following the work of [4].

In this paper, we define new mappings

$$\zeta_\varepsilon\colon W\Omega^\bullet_{k[\underline{X}]/k} \to \mathbb{R} \cup \{+\infty, -\infty\}\ ,$$

for all $\varepsilon > 0$ and prove that these are pseudovaluations. Moreover, we show that the set of all $x \in W\Omega^\bullet_{k[\underline{X}]/k}$, such that $\zeta_\varepsilon(x) \neq -\infty$, for some $\varepsilon > 0$, also define the overconvergent de Rham–Witt complex.

In order to do so, in the first section we recall the main results concerning the de Rham–Witt complex, especially in the case of a polynomial algebra.

The second section, which is the most technical one, consists of computations of products of specific elements of the de Rham–Witt complex. The results are explicit and proven in the case where k is any commutative $\mathbb{Z}_{\langle p \rangle}$-algebra. This enables us in the last section to define the pseudovaluations and prove that in the case of a perfect field of positive characteristic, we retrieve with these functions the overconvergent de Rham–Witt complex.

The product formula comes in handy to control the overconvergence of sequences defined by recursion. This is the main motivation for this work, which will allow us in subsequent papers to study the structure of the overconvergent de Rham–Witt complex and, eventually, to give an interpretation of F-isocrystals as overconvergent de Rham–Witt connections.

1. The de Rham–Witt complex for a polynomial ring

Let p be a prime number. Let k be a commutative $\mathbb{Z}_{\langle p \rangle}$-algebra. Throughout this article, for any $i, j \in \mathbb{N}$, we shall write:

$$[\![i, j]\!] := \mathbb{N} \cap [i, j] \,.$$

Let $n \in \mathbb{N}$ and write $k[\underline{X}] := k[X_1, \ldots, X_n]$. We will first recall basic properties of the de Rham–Witt complex of $k[\underline{X}]$, denoted $\left(W\Omega^{\bullet}_{k[\underline{X}]/k}, d\right)$ (for an introduction, see [5] or [6]). In degree zero, $W\Omega^0_{k[\underline{X}]/k}$ is isomorphic as a $W(k)$-algebra to $W(k[\underline{X}])$, the ring of Witt vectors over $k[\underline{X}]$.

There is a morphism of graded rings $F\colon W\Omega^{\bullet}_{k[\underline{X}]/k} \to W\Omega^{\bullet}_{k[\underline{X}]/k}$ called the **Frobenius endormorphism**, a morphism $V\colon W\Omega^{\bullet}_{k[\underline{X}]/k} \to W\Omega^{\bullet}_{k[\underline{X}]/k}$ of graded groups called the ***Verschiebung*** **endormorphism**, as well as a morphism of monoids $[\bullet] : (k[\underline{X}]\,, \times) \to (W(k[\underline{X}])\,, \times)$ called the **Teichmüller lift** such that:

$$\forall r \in k[\underline{X}]\,,\ F([r]) = [r^p]\,, \tag{1}$$

$$\forall m \in \mathbb{N},\ \forall x, y \in W\Omega^{\bullet}_{k[\underline{X}]/k},\ V^m(xF^m(y)) = V^m(x)\, y, \tag{2}$$

$$\forall m \in \mathbb{N},\ \forall x \in W\Omega^{\bullet}_{k[\underline{X}]/k},\ d(F^m(x)) = p^m F^m(d(x))\,, \tag{3}$$

$$\forall m \in \mathbb{N},\ \forall P \in k[\underline{X}]\,,\ F^m(d([P])) = \left[P^{p^m - 1}\right] d([P])\,, \tag{4}$$

$$\begin{gathered}\forall i, j \in \mathbb{N},\ \forall x \in W\Omega^{i}_{k[\underline{X}]/k},\ \forall y \in W\Omega^{j}_{k[\underline{X}]/k}, \\ d(xy) = (-1)^i\, xd(y) + (-1)^{(i+1)j}\, yd(x)\,,\end{gathered} \tag{5}$$

$$\begin{gathered}\forall m \in \mathbb{N},\ \forall (x_i)_{i \in [\![1,m]\!]} \in (W(k[\underline{X}]))^m\,, \\ d\left(\prod_{i=1}^{m} x_i\right) = \sum_{i=1}^{m} \left(\prod_{j \in [\![1,m]\!] \smallsetminus \{i\}} x_j\right) d(x_i)\,.\end{gathered} \tag{6}$$

We shall introduce basic elements on the de Rham–Witt complex, called basic Witt differentials, and recall how any de Rham–Witt differential on $k[\underline{X}]$ can be expressed as a series using these elements. We mostly follow [6].

Definition 1.1. — A **weight function** is a mapping $a\colon [\![1,n]\!] \to \mathbb{N}\left[\frac{1}{p}\right]$; for all $i \in [\![1,n]\!]$, its values shall be written as a_i. We define:

$$|a| \coloneqq \sum_{i=1}^{n} a_i,$$

and:

$$\underline{X}^a \coloneqq \prod_{i=1}^{n} {X_i}^{a_i}.$$

For any weight function a and any $J \subset [\![1,n]\!]$, we denote by $a|_J$ the weight function that for all $i \in [\![1,n]\!]$ satisfies:

$$a|_J(i) = \begin{cases} a_i & \text{if } i \in J, \\ 0 & \text{otherwise.} \end{cases}$$

The **support** of a weight function a is the following set:

$$\operatorname{Supp}(a) \coloneqq \{i \in [\![1,n]\!] \mid a_i \neq 0\}.$$

A **partition** I of a weight function a is a subset $I \subset \operatorname{Supp}(a)$. Its **size** is its cardinal. We will denote by $\mathcal{P}$ the set of all pairs (a, I), where a is a weight function, and I is a partition of a.

In all this paper, the p-adic valuation shall be denoted v_p. For a weight function a, we fix the following total order $\preceq$ on $\operatorname{Supp}(a)$:

$$\forall i, i' \in \operatorname{Supp}(a)\,,\ i \preceq i' \iff ((\mathrm{v}_p(a_i) \leqslant \mathrm{v}_p(a_{i'})) \wedge ((\mathrm{v}_p(a_i) = \mathrm{v}_p(a_{i'})) \implies (i \leqslant i'))).$$

We will denote by $\prec$ the associated strict total order and we also let $\min(a) \in \operatorname{Supp}(a)$ be the only element such that $\min(a) \preceq i$, for any $i \in \operatorname{Supp}(a)$.

Let $m \in [\![0,n]\!]$. Let $I \coloneqq \{i_j\}_{j \in [\![1,m]\!]}$ be a partition of a weight function a. We will always suppose that $i_j \prec i_{j'}$, for all $j, j' \in [\![1,m]\!]$, such that $j < j'$. By convention, we will say that $i_0 \preceq i$ and $i \prec i_{m+1}$ whenever $i \in \operatorname{Supp}(a)$. We define the following $m+1$ subsets of $\operatorname{Supp}(a)$ for any $l \in [\![0,m]\!]$:

$$I_l \coloneqq \{i \in \operatorname{Supp}(a) \mid i_l \preceq i \prec i_{l+1}\}.$$

Let a be a weight function. We set:

$$\mathrm{v}_p(a) \coloneqq \min\{\mathrm{v}_p(a_i) \mid i \in [\![1,n]\!]\},$$
$$u(a) \coloneqq \max\{0, -\mathrm{v}_p(a)\}.$$

If a is not the zero function, we put:

$$g(a) := F^{u(a)+\mathrm{v}_p(a)}\Big(d\Big(V^{u(a)}\Big(\Big[\underline{X}^{p^{-\mathrm{v}_p(a)}a}\Big]\Big)\Big)\Big).$$

Furthermore, if I is a partition of a, and η is any element in $W(k)$, we set:

(7)

$$e(\eta, a, I) := \begin{cases} d\Big(V^{u(a)}\Big(\eta\Big[\underline{X}^{p^{u(a)}a|_{I_0}}\Big]\Big)\Big)\prod_{l=2}^{\#I} g(a|_{I_l}) & \text{if } I_0 = \emptyset \text{ and } u(a) \neq 0, \\ V^{u(a)}\Big(\eta\Big[\underline{X}^{p^{u(a)}a|_{I_0}}\Big]\Big)\prod_{l=1}^{\#I} g(a|_{I_l}) & \text{otherwise.} \end{cases}$$

When $I_0 = \emptyset$ and $u(a) = 0$, then $V^{u(a)}\Big(\eta\Big[\underline{X}^{p^{u(a)}a|_{I_0}}\Big]\Big) = \eta$. So one can notice that, if one ignores η, the element defined above is a product of $\#I$ factors whenever $I_0 = \emptyset$, and of $\#I + 1$ factors otherwise, the factors being the images of an element through one of the functions d, g or V. We will use this fact later, when we define the pseudovaluations on the de Rham–Witt complex of a polynomial ring.

We recall the action of d, V and F on these elements.

Proposition 1.2. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$d(e(\eta, a, I)) = \begin{cases} 0 & \text{if } I_0 = \emptyset, \\ e(\eta, a, I \cup \{\min(a)\}) & \text{if } I_0 \neq \emptyset \text{ and } \mathrm{v}_p(a) \leqslant 0, \\ p^{\mathrm{v}_p(a)} e(\eta, a, I \cup \{\min(a)\}) & \text{if } I_0 \neq \emptyset \text{ and } \mathrm{v}_p(a) > 0. \end{cases}$$

Proof. — See [6, proposition 2.6]. □

Proposition 1.3. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$F(e(\eta, a, I)) = \begin{cases} e(\eta, pa, I) & \text{if } \mathrm{v}_p(a) < 0 \text{ and } I_0 = \emptyset, \\ e(p\eta, pa, I) & \text{if } \mathrm{v}_p(a) < 0 \text{ and } I_0 \neq \emptyset, \\ e(F(\eta), pa, I) & \text{if } \mathrm{v}_p(a) \geqslant 0. \end{cases}$$

Proof. — See [6, proposition 2.5]. □

Proposition 1.4. — *For any $(a, I) \in \mathcal{P}$ and any $\eta \in W(k)$, we have:*

$$V(e(\eta, a, I)) = \begin{cases} e\Big(V(\eta), \frac{a}{p}, I\Big) & \text{if } \mathrm{v}_p(a) > 0, \\ e\Big(p\eta, \frac{a}{p}, I\Big) & \text{if } \mathrm{v}_p(a) \leqslant 0 \text{ and } I_0 = \emptyset, \\ e\Big(\eta, \frac{a}{p}, I\Big) & \text{if } \mathrm{v}_p(a) \leqslant 0 \text{ and } I_0 \neq \emptyset. \end{cases}$$

Proof. — See [6, proposition 2.5]. □

The de Rham–Witt complex is endowed with a topology called the standard topology [5, I. 3.1.]. In this article, it will not be necessary to recall its definition, as we will only need the fact that a series of the form $\sum_{(a,I)\in\mathcal{P}} e(\eta_{a,I}, a, I)$,