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C^0 STABILITY OF BOUNDARY ACTIONS AND INEQUIVALENT ANOSOV FLOWS

BY Jonathan BOWDEN AND Kathryn MANN

ABSTRACT. – We give a topological stability result for the action of the fundamental group of a compact manifold of negative curvature on its boundary at infinity: any nearby action of this group by homeomorphisms of the sphere is semi-conjugate to the standard boundary action. Using similar techniques we prove a global rigidity result for the "slithering actions" of 3-manifold groups that come from skew-Anosov flows. As applications, we construct hyperbolic 3-manifolds that admit arbitrarily many topologically inequivalent Anosov flows, answering a question from Kirby's problem list, and also give a more conceptual proof of a theorem of the second author on *global* C^0-rigidity of geometric surface group actions on the circle.

RÉSUMÉ. – Le groupe fondamental d'une variété compacte agit sur le bord à l'infini de son revêtement universel. Nous démontrons un théorème de rigidité topologique pour cette famille d'actions: toute action suffisament proche de cette action standard est semi-conjuguée à celle-ci. Avec la même stratégie de preuve, nous démontrons un théorème de rigidité global pour les actions sur le cercle d'une variété de dimension 3 avec un « slithering » de Thurston. Comme applications, nous construisons pour tout N strictement positif une variété hyperbolique de dimension 3 qui admet au moins N flots d'Anosov topologiquement inéquivalents. Cette construction donne une réponse positive à une question de Christy de la liste de Kirby. Nous donnons aussi une preuve plus conceptuelle d'un théorème de la deuxième auteure sur la rigidité globale des actions géométriques d'un groupe de surface sur le cercle.

1. Introduction

This paper proves two related rigidity results for group actions on manifolds, with applications to skew-Anosov flows. The first is a general local rigidity result for the *boundary action* of the fundamental group of a closed negatively curved manifold.

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1. Local rigidity of boundary actions. – A major historical motivation for the study of rigidity of group actions comes from the classical (Selberg-Calabi-Weil) rigidity of lattices in Lie groups. Perhaps the best known example is Calabi's original theorem that, for $n \geq 3$, the fundamental group of a compact, hyperbolic n-manifold is locally rigid as a lattice in $\mathrm{SO}(n,1)$, later extended to a global rigidity result by Mostow. From a geometric-topological viewpoint, it is natural to consider the action of $\mathrm{SO}(n,1)$ on the boundary sphere of the compactification of hyperbolic n-space (the universal cover of the manifold in question) and several modern proofs of Mostow rigidity pass through the study of this boundary action. See [18] for a broad introduction to the subject.

More generally, if M is a closed n-dimensional manifold of (variable) negative curvature, its universal cover $\widetilde{M}$ still admits a natural compactification by a visual *boundary sphere*, denoted $\partial_\infty \widetilde{M}$ and the action of $\pi_1 M$ on $\widetilde{M}$ by deck transformations extends to an action by homeomorphisms on $\partial_\infty \widetilde{M}$, which we call the *boundary action*. However, even if M is smooth, $\partial_\infty \widetilde{M}$ typically has no more than a Hölder C^0 structure. This presents a new challenge for dynamicists, as many tools in rigidity theory originate either from hyperbolic smooth dynamics or the homogeneous (Lie group) setting, where differentiability plays an essential role.

As we will later show, in the C^0 context the best rigidity result one can hope for is *topological stability*. An action $\rho' : \Gamma \to \mathrm{Homeo}(X)$ of a group Γ on a space X is said to be a *topological factor* of an action $\rho : \Gamma \to \mathrm{Homeo}(Y)$ if there is a surjective, continuous map $h : X \to Y$ (called a semiconjguacy) such that $h \circ \rho' = \rho \circ h$. A group action $\rho : \Gamma \to \mathrm{Homeo}(X)$ is *topologically stable* if any action which is close to ρ in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(Y))$ is a factor of ρ. (1) Here and in what follows, $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(Y))$ is always equipped with the standard compact-open topology. Our first result is the following.

THEOREM 1.1 (Topological stability). – *Let M be a compact, orientable n-manifold with negative curvature, and $\rho_0 : \pi_1 M \to \mathrm{Homeo}(S^{n-1})$ the natural boundary action on $\partial_\infty \widetilde{M}$. There exists a neighborhood of ρ_0 in $\mathrm{Hom}(\pi_1 M, \mathrm{Homeo}(S^{n-1}))$ consisting of representations which are topological factors of ρ_0.*

Moreover, this topological stability is strong *in the following sense: for any neighborhood U of the identity in the space of continuous self-maps of S^{n-1}, there exists a neighborhood V of ρ_0 in $\mathrm{Hom}(\pi_1 M, \mathrm{Homeo}(S^{n-1}))$ so that every element of V is semi-conjugate to ρ_0 by some map in U.*

The statement of Theorem 1.1 is similar in spirit to the extensions of the classical C^1-structural stability for Anosov (or more generally, Axiom A) systems to topological stability proved by Walters [47] and Nitecki [42] in the 1970s. However, we are working in the context of group actions rather than individual diffeomorphisms, and further, we do not assume any regularity of the original boundary action that is to be perturbed. Thus, our tools are by necessity fundamentally different.

(1) Elsewhere in the literature this is also referred to as *semi-stability* or *(topological) structural stability*, see [42] for some discussion of terminology.

In the context of stability properties of group actions Sullivan [44] characterized which subgroups of $\mathrm{PSL}(2,\mathbb{C})$ exhibit C^1-stability (2) and also remarked that stability holds more generally within the class of actions on metric spaces that are *expansive-hyperbolic*, a class of actions that includes boundary actions of fundamental groups of closed negatively curved manifolds. This program was worked out in detail only quite recently by Kapovich-Kim-Lee [36], who show that what they call *S-hyperbolic* actions (a weakening of Sullivan's expansivity-hyperbolicity condition) are stable under perturbation with respect to a Lipschitz topology. While S-hyperbolic actions represent a broader class than those studied here, in the specific case of boundary actions of fundamental groups of negatively curved manifolds our result is stronger and quite different in spirit. We do not aim to preserve "hyperbolic"-like behavior, and consider perturbations in the C^0-topology, which can be much more violent and introduce wandering domains. Thus, one can view Theorem 1.1 as a strict strengthening of Kapovich-Kim-Lee's topological stability for the restricted case of boundary actions.

Sharpness. – As hinted above, one cannot replace "factor of" with "conjugate to" in Theorem 1.1. In Section 4, we show that nearby, non-conjugate topological factors do occur for boundary actions of closed negatively curved manifolds. We give two sample constructions. One comes from *Cannon-Thurston* maps, special to the case where M is a hyperbolic 3-manifold, and the other is a general "blow-up" type construction, applicable to C^1 examples in all dimensions.

2. Global rigidity of slithering actions. – In the case where $\dim(M) = 2$, and hence $\partial_\infty(\widetilde{M}) = S^1$, a stronger global rigidity result for boundary actions of surface groups was proved by the second author in [38] (see also [8], [40]). Using the techniques of Theorem 1.1 we can recover this, and in fact generalize it to the broader context of group actions on S^1 arising from *slitherings* associated to skew-Anosov flows on 3-manifolds, in the sense of Thurston [46]. As we discuss in the next paragraph, these flows are basic examples in hyperbolic dynamics. Our rigidity result is the following.

THEOREM 1.2 (Global rigidity of skew-Anosov slithering actions).
Let $\mathcal{F}^s$ be the weak stable foliation of a skew-Anosov flow on a closed 3-manifold M, and $\rho_s : \pi_1 M \to \mathrm{Homeo}_+(S^1)$ the associated slithering action. Then the connected component of ρ_s in $\mathrm{Hom}(\pi_1 M, \mathrm{Homeo}_+(S^1))$ consists of representations "semi-conjugate" to ρ_s in the sense of Ghys [26].

Definitions and properties of skew-Anosov flows and slitherings are recalled in Section 5. Note that the notion of semi-conjugacy of circle maps in the statement above is *not* the same as in the definition of topological factor; unfortunately the terminology "semi-conjugacy" in this sense has also become somewhat standard. To avoid confusion, we will follow [48] and use the term *weak conjugacy* for this property of actions on the circle. It has also been referred to as "monotone equivalence" by Calegari.

A consequence of the above theorem is a new, independent proof of the main result of [38] on global C^0 rigidity of geometric surface group actions on S^1. See Corollary 5.12 below.

(2) Sullivan's main result is in the opposite direction to ours: he shows that, among subgroups of $\mathrm{PSL}(2,\mathbb{C})$, C^1-stability implies convex-cocompactness.