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SINGULARITY FORMATION FOR BURGERS' EQUATION WITH TRANSVERSE VISCOSITY

BY Charles COLLOT, Tej-Eddine GHOUL
AND Nader MASMOUDI

ABSTRACT. – We consider Burgers' equation with transverse viscosity

$$\partial_t u + u\partial_x u - \partial_{yy} u = 0, \ \ (x,y) \in \mathbb{R}^2, \ \ u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}.$$

We construct and describe precisely a family of solutions which become singular in finite time by having their gradient becoming unbounded. To leading order, the solution is given by a backward self-similar solution of Burgers' equation along the x variable, whose scaling parameters evolve according to parabolic equations along the y variable, one of them being the quadratic semi-linear heat equation. We develop a new framework adapted to this mixed hyperbolic/parabolic blow-up problem, revisit the construction of flat blow-up profiles for the semi-linear heat equation, and the self-similarity in singularities of the inviscid Burgers' equation.

RÉSUMÉ. – L'équation de Burgers avec viscosité transverse

$$\partial_t u + u\partial_x u - \partial_{yy} u = 0, \ \ (x,y) \in \mathbb{R}^2, \ \ u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$$

est considérée. Nous construisons et décrivons précisément une famille de solutions qui deviennent singulières en temps fini par la divergence de leur gradient. À l'ordre principal, la solution est donnée par un profil auto-similaire rétrograde de l'équation de Burgers dans la direction associée à la variable x, et dont les paramètres d'échelle évoluent selon des équations paraboliques dans la direction associée à la variable y, l'une d'elle étant l'équation de la chaleur semi-linéaire quadratique. Nous développons de nouvelles techniques pour ce problème d'explosion mixte hyperbolique et parabolique, revisitons la construction de solutions explosives plates pour l'équation de la chaleur semi-linéaire, et l'auto-similarité des singularités pour l'équation de Burgers non visqueuse.

1. Introduction

1.1. Setting of the problem and motivations

We consider Burgers' equation with transverse viscosity

$$\begin{cases} \partial_t u + u\partial_x u - \partial_{yy} u = 0, \ \ (x,y) \in \mathbb{R}^2, \\ u_{t=0} = u_0, \end{cases} \tag{1.1}$$

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for $u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$. The present study is motivated by the following. This model reduces to the classical inviscid Burgers' equation for solutions that are independent of the transverse variable $u(t,x,y) = U(t,x)$, which is a classical example of a nonlinear hyperbolic equation for which initially smooth solutions can become singular in finite time, see for example [11, 27]. The effects of viscosity in the streamwise direction, namely the equation $\partial_t u + u\partial_x u - \epsilon\partial_{xx} u = 0$, have been extensively studied, see [18, 19] and references therein. This work aims at understanding precisely the consequence of an additional effect, here the transverse viscosity, on a blow-up dynamics that it does not prevent. Moreover, this new effect changes the nature of the equation which is of a mixed hyperbolic/parabolic type. Handling these two issues, our result then extends known ones for blow-ups in a new direction, as well as raising new interesting problems, see the comments after the main Theorem 3.

More importantly, the study of (1.1) is motivated by fluid dynamics, from the fact that it is a simplified version of Prandtl's boundary layer equations. Solutions to Prandtl's equations might blow up in finite time [5, 12, 21] but a precise description of the singularity formation is still lacking. The present work is a step towards that goal and this issue will be investigated in a forthcoming work. Finally, let us mention that there has been recent progress on other models for singular solutions in fluid dynamics, see [9, 6, 28] and references therein.

The existence of smooth enough solutions to (1.1) follows from classical arguments. For example, relying on a fixed point argument and energy estimates, one can show that the equation is locally well-posed in $H^s(\mathbb{R}^2)$ for $s \geq 3$. There then holds the following blow-up criterion (again from energy estimates because of the identity $|\int uvv_x| \lesssim \|u_x\|_{L^\infty} \int v^2$): the solution u blows up at time $T > 0$ if and only if

$$\limsup_{t\uparrow T} \|\partial_x u\|_{L^\infty(\mathbb{R}^2)} = +\infty. \tag{1.2}$$

The existence of global kinetic solutions $u \in L^\infty([0,+\infty), L^1(\mathbb{R}^2))$ has been showed by Chen and Perthame [8] following the framework of Lions, Perthame and Tadmor [22]. We refer to [27] for an introduction on kinetic solutions for scalar conservation laws. We expect singularities for such low regularity solutions to be different than the solutions in the present paper, as regularity plays a key role in the blow-up mechanism we describe. Before stating the main theorem, let us give the structure of the singularities of Burgers' equation, and of the ones of the parabolic system encoding the effects of the transverse viscosity.

1.2. Self-similarity in shocks for Burgers' equation

Burgers' equation

$$\begin{cases} \partial_t U + U\partial_x U = 0, & x \in \mathbb{R}, \\ U_{t=0} = U_0 \end{cases} \tag{1.3}$$

admits solutions becoming singular in finite time in a self-similar way:

$$U(t,x) = \mu^{-1}(T-t)^{\frac{1}{2i}}\Psi_i\left(\mu\frac{x}{(T-t)^{1+\frac{1}{2i}}}\right),$$

where $(\Psi_i)_{i\in\mathbb{N}^*}$ is a family of analytic profiles (see [13] for example), and where $\mu > 0$ is a free parameter. They are at the heart of the shock formation, a fact that is rarely emphasized, which lead us to give a precise and concise study in Section 2. Self-similar and discretely self-similar blow-up profiles for Burgers' equation are classified in Proposition 4. Different scaling laws are thus possible, depending on the initial condition via its behavior near the characteristic where the shock will form, which has to do with the fact that the scaling group of (1.3) is two-dimensional, see Section (2). This possibility of several scaling exponents is referred to as self-similarity of the second kind [1]. For each $i \in \mathbb{N}^*$, Ψ_i, defined in Proposition 5, is an odd decreasing profile, which is nonnegative and concave on $(-\infty, 0]$ and such that $\partial_X \Psi_i$ is minimal at the origin with asymptotic $\Psi_i(X) = -X + X^{2i+1}$ as $X \to 0$. One has in particular the formula

$$\Psi_1(X) := \left(-\frac{X}{2} + \left(\frac{1}{27} + \frac{X^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + \left(-\frac{X}{2} - \left(\frac{1}{27} + \frac{X^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \tag{1.4}$$

for the fundamental one [7]. As in the above formula, all these profiles are unbounded at infinity but they emerge nonetheless from well localized initial data. A precise description of these profiles is given in Proposition 5. Any regular enough non-degenerate solution v to (1.3) that forms a shock at (T, x_0) is equivalent to leading order near the singularity to a self-similar profile Ψ_i up to the symmetries of the equation

$$U(t,x) \sim (T-t)^{\frac{1}{2i}} \mu^{-1} \Psi_i \left(\mu \frac{x - (x_0 - c(T-t))}{(T-t)^{1+\frac{1}{2i}}} \right) + c \quad \text{as } (t,x) \to (T, x_0), \tag{1.5}$$

see Proposition 9. The blow-up dynamics involving the concentration of Ψ_1 is a stable one for smooth enough solutions. The scenario corresponding to the concentration of Ψ_i for $i \geq 2$ is an unstable one. For a suitable topological functional space, the set of initial conditions leading to the concentration of Ψ_i for $i \geq 2$ is located at the boundary of the set of initial condition leading to the concentration of Ψ_1, and admits $2(i-1)$ instability directions yielding one or several shocks formed by Ψ_j for $j < i$. The linearised dynamics is described in Proposition 8.

1.3. Blow-up for the reduced parabolic system

For a solution u to (1.1) that is odd in x, the behavior on the transverse axis $\{x = 0\}$ is encoded by a closed system, which is the motivation for this symmetry assumption. It admits solutions blowing up simultaneously with a precise behavior. Indeed, assume $\partial_x^j u_0(0, y) = 0$ for all $y \in \mathbb{R}$ for $2 \leq j \leq 2i$ for some integer $i \in \mathbb{N}$. This remains true for later times and the trace of the derivatives

$$\xi(t,y) := -\partial_x u(t,0,y) \ \text{ and } \ \zeta(t,y) := \partial_x^{2i+1} u(t,0,y) \tag{1.6}$$

solve the parabolic system

$$\begin{cases} (NLH) \ \xi_t - \xi^2 - \partial_{yy}\xi = 0, \\ (LFH) \ \zeta_t - (2i+2)\xi\zeta - \partial_{yy}\zeta = 0. \end{cases} \tag{1.7}$$

Solutions to the nonlinear heat equation (NLH) might blow up in finite time, a dynamics that can be detailed precisely, see [26] for an overview. There exists a stable fundamental blow-up [2, 3, 17, 24], and a countable number of unstable "flatter" blow-ups [3, 15], all driven to leading order by the ODE $f' = f^2$. For the present work, we had to show additional