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CURVE TEST FOR ENHANCED IND-SHEAVES AND HOLONOMIC D-MODULES, II

BY TAKURO MOCHIZUKI

Dedicated to Professor Masaki Kashiwara on the occasion of his 75th birthday

ABSTRACT. – In this series of papers, we study a condition when a complex of cohomologically $\mathbb{R}$-constructible enhanced ind-sheaves K is induced by a cohomologically holonomic complex of D-modules. In this paper (Part II), we characterize such K in terms of the restriction of K to holomorphic curves, by using the results on multi-sets of subanalytic functions in the part I.

RÉSUMÉ. – Dans cette série d'articles, nous étudions une condition lorsqu'un complexe d'ind-faisceaux renforcés à cohomologie $\mathbb{R}$-constructibles K est induit par un complexe des D-modules à cohomologie holonomes. Dans ce document (partie II), nous caractérisons un tel K en termes de restriction de K aux courbes holomorphes, en utilisant les résultats sur des multiensembles de fonctions sous-analytiques dans la partie I.

1. Introduction

1.1. Main results

Let X be a complex manifold. Let $\mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_X)$ denote the derived category of cohomologically holonomic complexes of $\mathscr{D}_X$-modules. Let $\mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$ denote the derived category of $\mathbb{R}$-constructible enhanced ind-sheaves on X. D'Agnolo and Kashiwara constructed the enhanced de Rham functor $\mathrm{DR}^{\mathsf{E}}_X : \mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$, and they proved that $\mathrm{DR}^{\mathsf{E}}_X$ is fully faithful and compatible with standard 6-operations. (See [3]. See also useful surveys [7, 8].)

To study the essential image $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$, we introduce the following condition for objects $K \in \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$.

— Set $\Delta := \{|z| < 1\}$. Let $\varphi : \Delta \longrightarrow X$ be any holomorphic map. Then, $\mathsf{E}\varphi^{-1}(K) \in \mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_\Delta)$.

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The condition determines the full subcategory $\mathsf{E}^b_{\triangle}(\mathrm{I}\mathbb{C}_X) \subset \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_X)$. We shall prove the following theorem.

THEOREM 1.1 (Theorem 5.1). – $\mathsf{E}^b_{\triangle}(\mathrm{I}\mathbb{C}_X)$ *is equal to* $\mathsf{E}^b_{\mathscr{D}}(\mathrm{I}\mathbb{C}_X)$.

A holonomic $\mathscr{D}$-module can be locally described as the gluing of meromorphic flat connections on complex analytic subspaces. Hence, it is a key step to study such a characterization for meromorphic flat connections. Let H be a complex hypersurface of X. Let $\boldsymbol{X}(H)$ denote the bordered space $(X \setminus H, X)$ in the sense of [3, 4]. We obtain the full subcategory $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) \subset \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ of the objects obtained as $\mathrm{DR}^{\mathsf{E}}_{\boldsymbol{X}(H)}(V)[-n]$ for meromorphic flat connections (V, ∇) on (X, H).

We set $\Delta^* := \Delta \setminus \{0\}$. Let $\boldsymbol{X}(H)^{\boldsymbol{\Delta}(0)}$ be the set of holomorphic maps $\varphi : \Delta \to X$ satisfying $\varphi(\Delta^*) \subset X \setminus H$ and $\varphi(0) \in H$. We obtain the full subcategory $\mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) \subset \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ of the objects K satisfying the following conditions.

— $K_{|X\setminus H}$ is induced by a local system on $X \setminus H$.

— For any $\varphi \in \boldsymbol{X}(H)^{\boldsymbol{\Delta}(0)}$, $\mathsf{E}\varphi^{-1}(K)$ is an object in $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{\Delta}(0)})$.

Clearly, $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ is an object in $\mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$. The following is the essential part in the proof of Theorem 1.1.

THEOREM 1.2. – $\mathsf{E}_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)}) = \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$.

1.2. Summary of [24]

Recall that for a meromorphic flat bundle (V, ∇) on $(\Delta, 0)$, we obtain a multi-set of ramified irregular values $(\mathrm{Irr}(V, \nabla), \mathrm{rank})$ which is a multi-subset of $\mathscr{O}_{\Delta}(*0)^{(e)}_0 / \mathscr{O}^{(e)}_{\Delta,0}$ (See [24, §1.2] or §2.2 below.) For $K \in \mathsf{E}^b_{\mathrm{mero}}(\mathrm{I}\mathbb{C}_{\boldsymbol{\Delta}(0)})$, there exists a meromorphic flat bundle (V, ∇) on $(\Delta, 0)$ such that $K \simeq \mathrm{DR}^{\mathsf{E}}_{\boldsymbol{\Delta}(0)}(V)[-1]$. We set $(\mathrm{Irr}(K), \mathrm{rank}) := (\mathrm{Irr}(V, \nabla), \mathrm{rank})$.

As a preliminary for the proof of Theorem 1.2, we introduce the following condition for $K \in \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$ at $P \in H$.

(Condition 4.13): There exist a neighborhood X_P of P in X and a good multi-set of ramified irregular values $(\mathscr{I}_P, \mathfrak{m}_P)$ at P such that for any $\varphi \in \boldsymbol{X}_P(H_P)^{\boldsymbol{\Delta}(0)}$, we obtain $\mathrm{Irr}(\mathsf{E}\varphi^{-1}(K), \mathrm{rank}) = \varphi^*(\mathscr{I}_P, \mathfrak{m}_P)$.

We proved the following proposition in [24].

PROPOSITION 1.3. – *For any* $K \in \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$, *there exists an at most* $(\dim_{\mathbb{R}} H - 2)$-*dimensional closed subanalytic subset* $Z \subset H$ *such that* (i) Z *contains the singular locus of* H, (ii) *Condition* 4.13 *is satisfied for* K *at any point of* $H \setminus Z$. □

We also proved the following theorem in [24].

THEOREM 1.4. – *Suppose* $\dim_{\mathbb{C}} X = 2$. *Then, for any* $K \in \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$, *there exists a proper morphism of complex manifolds* $\Psi : X_1 \longrightarrow X$ *such that* (i) $H_1 := \Psi^{-1}(H)$ *is normal crossing,* (ii) $X_1 \setminus H_1 \simeq X \setminus H$, (iii) *Condition* 4.13 *is satisfied for* $\mathsf{E}\Psi^{-1}(K)$ *at any points of* H_1. □

1.3. Outline of the proof of Theorem 1.2

1.3.1. *Construction of good meromorphic flat bundles.* – Set $X = \Delta^2$ and $H = \{z_1 = 0\}$. Let $K \in \mathsf{E}_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$. Let $(\mathscr{J}, \mathfrak{m})$ be a good multi-set of irregular values on (X, H). Suppose that Condition 4.13 is satisfied for K and $(\mathscr{J}, \mathfrak{m})$ at any point of H, i.e., for any $\varphi \in \boldsymbol{X}(H)^{\boldsymbol{\Delta}(0)}$, $(\mathrm{Irr}(\mathsf{E}\varphi^{-1}(K)), \mathrm{rank}) = \varphi^*(\mathscr{J}, \mathfrak{m})$ holds. Let us explain an outline of the proof of Theorem 1.2 in this situation.

Let $\varphi_0 : \Delta \longrightarrow X$ be defined by $\varphi_0(z) = (z, 0)$. Because $K \in \mathsf{E}^b_{\circledcirc}(\mathrm{I}\mathbb{C}_{\boldsymbol{X}(H)})$, there exists a meromorphic flat bundle (V_0, ∇_0) on $(\Delta, 0)$ with an isomorphism $\mathrm{DR}^{\mathsf{E}}_{\boldsymbol{\Delta}(0)}(V)[-1] \simeq \mathsf{E}\varphi_0^{-1}(K)$. By using Theorem 2.8, we obtain a good meromorphic flat bundle (V, ∇) on (X, H) such that (i) $\varphi_0^*(V, \nabla) \simeq (V_0, \nabla_0)$, (ii) the multi-set of irregular values of (V, ∇) is $(\mathscr{J}, \mathfrak{m})$. (See [24, §1.2] or §2.2 below for the notion of good meromorphic flat bundles.) Clearly, there exists a natural isomorphism $\Phi : K_{|X \setminus H} \simeq \mathrm{DR}^{\mathsf{E}}_{X \setminus H}(V_{|X \setminus H})[-2]$. We would like to prove that Φ extends to an isomorphism $K \simeq \mathrm{DR}^{\mathsf{E}}_{\boldsymbol{X}(H)}(V)[-2]$.

1.3.2. *Prolongations of local systems.* – As a preliminary for the further argument, we shall study a more general type of $\mathbb{R}$-constructible enhanced ind-sheaves. Let $\boldsymbol{M} = (M, \check{M})$ be a subanalytic bordered space in the sense of [3, 4]. By setting $\mathbb{I} := \{0 \leq t < 1\}$ and $\mathbb{I}^{\circ} := \mathbb{I} \setminus \{0\}$, we obtain the bordered space $\boldsymbol{\mathbb{I}}^{\circ} := (\mathbb{I}^{\circ}, \mathbb{I})$. Let L be a local system of rank r on M. We shall study $K \in \mathsf{E}^b_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\boldsymbol{M}})$ with an isomorphism $\Phi : K_{|M} \simeq L$ satisfying the following condition.

(Condition 3.39): Let $\gamma : \boldsymbol{\mathbb{I}}^{\circ} \longrightarrow \boldsymbol{M}$ be any morphism of the subanalytic bordered spaces. Then, there exist a tuple of subanalytic functions $g_1, \ldots, g_r$ on $\boldsymbol{\mathbb{I}}^{\circ}$ and an isomorphism $\mathsf{E}\gamma^{-1}(K) \simeq \bigoplus_{i=1}^r \mathbb{C}^{\mathsf{E}}_{\boldsymbol{\mathbb{I}}^{\circ}} \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}$.

Such (K, Φ) is called a prolongation of L. We shall prove the following.

Proposition 1.5 (Proposition 3.43). – *Let (K_i, Φ_i) $(i = 1, 2)$ be prolongations of L as above. Then, the isomorphism $\Phi_2^{-1} \circ \Phi_1 : K_1|_{X \setminus H} \simeq K_2|_{X \setminus H}$ extends to an isomorphism $K_1 \simeq K_2$ if and only if the following holds.*

— *For any morphism $\gamma : \boldsymbol{\mathbb{I}}^{\circ} \longrightarrow \boldsymbol{M}$, the isomorphism $\gamma|_{\mathbb{I}^{\circ}}^{-1}\big(\Phi_2^{-1} \circ \Phi_1\big)$ extends to an isomorphism $\mathsf{E}\gamma^{-1}(K_1) \simeq \mathsf{E}\gamma^{-1}(K_2)$.*

Let $\mathrm{Sub}(\boldsymbol{M})$ denote the space of subanalytic functions on $\boldsymbol{M}$. We obtain $\overline{\mathrm{Sub}}(\boldsymbol{M})$ as the quotient of $\mathrm{Sub}(\boldsymbol{M})$ divided by the subspace of the locally bounded functions. There exists a naturally defined partial order $\prec$ on $\overline{\mathrm{Sub}}(\boldsymbol{M})$.

In Condition 3.39, for γ, we obtain a multi-subset $(I_\gamma, \mathfrak{m}_\gamma)$ of $\overline{\mathrm{Sub}}(\boldsymbol{\mathbb{I}}^{\circ})$ induced by the tuple $g_1, \ldots, g_r$, and the isomorphism $\mathsf{E}\gamma^{-1}(K) \simeq \bigoplus_{i=1}^r \mathbb{C}^{\mathsf{E}}_{\boldsymbol{\mathbb{I}}^{\circ}} \overset{+}{\otimes} \mathbb{C}_{t \geq g_i}$ induces a filtration $\mathscr{F}$ on $\mathsf{E}\gamma^{-1}(K)$ indexed by $(I_\gamma, \mathfrak{m}_\gamma)$. In particular, we obtain a filtration $\mathscr{F}$ on $\gamma^{-1}|_{\mathbb{I}^{\circ}}(L)$. We can recover $\mathsf{E}\gamma^{-1}(K)$ from $\gamma^{-1}|_{\mathbb{I}^{\circ}}(L)$ with the filtration $\mathscr{F}$ and the index set $(I_\gamma, \mathfrak{m}_\gamma)$. The following is a special case of Proposition 3.51.

Proposition 1.6. – *Suppose that $\check{M}$ is a real analytic manifold with smooth boundary, that M is the interior part of $\check{M}$, and that M and $\partial\boldsymbol{M} := \check{M} \setminus M$ are simply connected. Let K be a prolongation of L. We assume that there exists a multi-subset $(\mathscr{J}, \mathfrak{m})$ of $\overline{\mathrm{Sub}}(\boldsymbol{M})$ such that the following condition is satisfied.*

— *For any morphism $\gamma : \boldsymbol{\mathbb{I}}^{\circ} \longrightarrow \boldsymbol{M}$ such that $\gamma(0) \in \partial M$, we obtain $(I_\gamma, \mathfrak{m}_\gamma) = \gamma^*(\mathscr{J}, \mathfrak{m})$.*