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AFFINE LAUMON SPACES AND A CONJECTURE OF KUZNETSOV

BY Andrei NEGUȚ

ABSTRACT. – We prove a conjecture of Kuznetsov stating that the equivariant K-theory of affine Laumon spaces is the universal Verma module of the quantum affine algebra of $U_q(\dot{\mathfrak{gl}}_n)$. We do so by reinterpreting the well-known action of the quantum toroidal algebra of type $U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$ on the K-theory of affine Laumon spaces in terms of the shuffle algebra, which allows us to use a certain embedding of the quantum affine algebra into the quantum toroidal algebra.

RÉSUMÉ. – Nous prouvons une conjecture de Kuznetsov qui affirme que la K-théorie équivariante des espaces affines de Laumon est le module universel de Verma de l'algèbre quantique affine de $U_q(\dot{\mathfrak{gl}}_n)$. Nous le faisons en réinterprétant l'action bien connue de l'algèbre quantique toroïdale de type $U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$ sur la K-théorie des espaces affines de Laumon en termes d'algèbre de battage, ce qui nous permet d'utiliser une certaine incorporation de l'algèbre quantique affine dans l'algèbre quantique toroïdale.

1. Introduction

Laumon spaces for the group GL_n parametrize flags of torsion-free sheaves on $\mathbb{P}^1$:

$$\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{n-1} \subset \mathcal{O}_{\mathbb{P}^1}^{\oplus n}, \tag{1.1}$$

whose fibers near $\infty \subset \mathbb{P}^1$ match a fixed full flag of subspaces of $\mathbb{C}^n$. Laumon spaces are disconnected, with components indexed by vectors $\mathbf{d} = (d_1, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$ that keep track of the first Chern classes of the sheaves in (1.1). The component indexed by $\mathbf{d}$ coincides with the space of framed degree $\mathbf{d}$ quasimaps into the complete flag variety, hence the interest in these objects in representation theory.

We will denote quantum affine and quantum toroidal algebras by $U_q(\dot{\mathfrak{gl}}_n)$, $U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$, respectively (we use dots instead of the more customary hats in order to avoid double hats on our symbols). The equivariant K--theory groups of Laumon spaces have been studied and

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identified with the universal Verma module of $U_q(\mathfrak{sl}_n)$ in [1]. In the present paper, we study an "affine" (1) version of these spaces, denoted by:

$$\mathcal{M} = \bigsqcup_{\mathbf{d}=(d_1,\dots,d_n)\in\mathbb{N}^n} \mathcal{M}_{\mathbf{d}},$$

whose definition we will recall in Subsection 3.1. The reference [1] constructs an action of $U_q(\dot{\mathfrak{sl}}_n) \curvearrowright K = K_{\text{equiv}}(\mathcal{M})$ and recalls a conjecture of Kuznetsov that this action can be extended to $U_q(\dot{\mathfrak{gl}}_n) \curvearrowright K$. Our main purpose is to prove this fact:

THEOREM 1.1. – *There is a geometric action of the affine quantum algebra $U_q(\dot{\mathfrak{gl}}_n)$ on K, with the latter being isomorphic to the universal Verma module.*

Affine Laumon spaces appear naturally in mathematical physics, geometry and representation theory as semismall resolutions of singularities of Uhlenbeck spaces for $\dot{\mathfrak{gl}}_n$. In [15], we use Theorem 1.1 to prove a conjecture of Braverman that relates the Nekrasov partition function of $\mathcal{N} = 2$ supersymmetric $U(n)$ gauge theory with bifundamental matter in the presence of a complete surface operator to the elliptic Calogero-Moser system. The fact that the K-theory group of $\mathcal{M}$ is isomorphic to the universal Verma module of $U_q(\dot{\mathfrak{gl}}_n)$ is crucial to our proof.

Reference [16] constructed an action of the bigger algebra $U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$ on K, by generators and relations. We will recast this action in terms of the shuffle algebra realization of $U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$, see [14]. Thus, any element of the shuffle algebra gives rise to an operator on K. In particular, we have introduced in *loc. cit.* the elements:

$$S_m^{\pm}, T_m^{\pm} \in U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n) \Rightarrow S_m^{\pm}, T_m^{\pm} \curvearrowright K \tag{1.2}$$

for any Laurent polynomial $m(z_i,\dots,z_{j-1})$ and any pair of integers $i < j$. When m is the constant Laurent polynomial 1, the operators (1.2) will give rise to the action of the root generators of $U_q(\dot{\mathfrak{gl}}_n)$ on K, and we will use this fact to prove Theorem 1.1.

The word "geometric" in the statement of Theorem 1.1 comes from the fact that the operators (1.2) will be given by certain explicit correspondences. Specifically, we will define three types of correspondences in Section 4:

— the fine correspondences $\mathfrak{Z}_{[i;j)}$ in Definition 4.3;

— the eccentric correspondences $\overline{\mathfrak{Z}}_{[i;j)}$ in Definition 4.7;

— the smooth correspondences $\mathfrak{W}_{(k,\dots,k)}$ in (4.37).

The fine and eccentric correspondences are equipped with tautological line bundles $\mathcal{L}_i,\dots,\mathcal{L}_{j-1}$. With this in mind, we may identify the operators (1.2) with those induced by the fine and eccentric correspondences:

THEOREM 1.2. – *The shuffle element $S_m^{\pm}$ (resp. $T_m^{\pm}$) $\in U_{q,\overline{q}}(\ddot{\mathfrak{gl}}_n)$ acts on K via:*

$$m\left(\mathcal{L}_i,\dots,\mathcal{L}_{j-1}\right) \quad \text{on} \quad \mathfrak{Z}_{[i;j)} \ \ (\textit{resp. } \overline{\mathfrak{Z}}_{[i;j)})$$

interpreted as an operator on K in (4.18) (*resp.* (4.19)). *Similarly, the shuffle element $G_{\pm(k,\dots,k)}$ of* (2.20) *acts on K via the smooth correspondence $\mathfrak{W}_{(k,\dots,k)}$.*

(1) The word "affine" here does not refer to geometric properties of the spaces in question, but it refers to the fact that their cohomology/K-theory groups are controlled by $\widehat{\mathfrak{gl}}_n$ instead of $\mathfrak{sl}_n$.

The content of the present paper and of [14] is synthesized in [13], with additional details, in the related context when affine Laumon spaces are replaced by Nakajima cyclic quiver varieties. Indeed, affine Laumon spaces may be interpreted as "chainsaw quiver varieties" (see [7]), which we review in Subsection 3.8.

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2. The quantum toroidal and shuffle algebras

2.1. – Let us now review the main constructions of [14], and introduce certain notations and results that will be used. In the present paper, we will encounter variables:

$$x_{i1}, x_{i2}, \dots$$

for arbitrary $i \in \{1, \dots, n\}$. The first index is called the *color* of the variable x_{ia}, and is denoted by $i = \text{col} x_{ia}$. Although colors come from the set $\{1, \dots, n\}$, in many of our formulas we will encounter arbitrary colors $i \in \mathbb{Z}$, by the convention:

$$\Big(z \text{ of color } i\Big) \quad \text{is identified with} \quad \Big(z\overline{q}^{-2\lfloor \frac{i-1}{n} \rfloor} \text{ of color } \bar{\imath}\Big), \tag{2.1}$$

where $\bar{\imath}$ denotes the residue class of $i \in \mathbb{Z}$ in the set $\{1, \dots, n\}$. As an example, let us consider the following color-dependent rational function:

$$\zeta\left(\frac{z}{w}\right) = \left(\frac{zq\overline{q}^{2\lceil \frac{i-j}{n} \rceil} - wq^{-1}}{z\overline{q}^{2\lceil \frac{i-j}{n} \rceil} - w}\right)^{\delta^0_{i-j \bmod n} - \delta^0_{i-j+1 \bmod n}} \tag{2.2}$$

for variables z, w of colors $i, j \in \mathbb{Z}$. If we wanted to convert the right-hand sides of (2.2) into an expression that only involves variables of colors $\bar{\imath}, \bar{\jmath} \in \{1, \dots, n\}$, then:

$$\zeta\left(\frac{z}{w}\right) = \begin{cases} \frac{zq - wq^{-1}}{z-w} & \text{if } \bar{\imath} = \bar{\jmath} \\ \frac{z-w}{zq-wq^{-1}} & \text{if } \bar{\imath}+1 = \bar{\jmath} \\ \frac{z\overline{q}^2 - w}{zq\overline{q}^2 - wq^{-1}} & \text{if } \bar{\imath} = n, \bar{\jmath} = 1 \\ 1 & \text{otherwise.} \end{cases} \tag{2.3}$$

2.2. – Let us now introduce the trigonometric shuffle algebra of type $\widehat{A}_n$ (see [5] for the original inspiration for shuffle algebras in the context of elliptic quantum algebras of finite type). A rational function:

$$R(\dots, z_{i1}, \dots, z_{ik_i}, \dots) \tag{2.4}$$

(i goes from 1 to n) will be called *color-symmetric* if it is symmetric in $z_{i1}, \dots, z_{ik_i}$ for all i separately. We will refer to the vector $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ as the degree of R, and we will write $\mathbf{k}! = k_1! \cdots k_n!$. Let $\mathbb{F} = \mathbb{Q}(q, \overline{q})$ and define the vector space:

$$\mathcal{V} = \bigoplus_{\mathbf{k} \in \mathbb{N}^n} \mathbb{F}(\dots, z_{i1}, \dots, z_{ik_i}, \dots)^{\text{Sym}}_{1 \le i \le n}, \tag{2.5}$$