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TORSORS ON LOOP GROUPS AND THE HITCHIN FIBRATION

BY Alexis BOUTHIER AND Kęstutis ČESNAVIČIUS

Abstract. – In his proof of the fundamental lemma, Ngô established the product formula for the Hitchin fibration over the anisotropic locus. One expects this formula over the larger generically regular semisimple locus, and we confirm this by deducing the relevant vanishing statement for torsors over $R(\!(t)\!)$ from a general formula for $\mathrm{Pic}(R(\!(t)\!))$. In the build up to the product formula, we present general algebraization, approximation, and invariance under Henselian pairs results for torsors, give short new proofs for the Elkik approximation theorem and the Chevalley isomorphism $\mathfrak{g}/\!\!/G \cong \mathfrak{t}/W$, and improve results on the geometry of the Chevalley morphism $\mathfrak{g} \to \mathfrak{g}/\!\!/G$.

Résumé. – Dans sa preuve du lemme fondamental, Ngô établit une formule du produit au-dessus du lieu anisotrope. On s'attend à ce qu'une telle formule s'étende au-dessus de l'ouvert génériquement régulier semisimple. Nous établissons cette formule en la déduisant d'un résultat d'annulation de torseurs sous des groupes de lacets à partir d'une formule générale pour $\mathrm{Pic}(R(\!(t)\!))$. Au cours de la preuve, nous montrons des résultats généraux d'algébrisation, d'approximation et d'invariance hensélienne pour des torseurs ; nous donnons de nouvelles preuves plus concises du théorème d'algébrisation d'Elkik et de l'isomorphisme de Chevalley $\mathfrak{g}/\!\!/G \cong \mathfrak{t}/W$ et améliorons les énoncés sur la géométrie du morphisme de Chevalley $\mathfrak{g} \to \mathfrak{g}/\!\!/G$.

1. Introduction

1.1. The product formula for the Hitchin fibration

A key insight in Ngô's proof of the fundamental lemma in [71] is to relate the affine Springer fibration, which over an equicharacteristic local field geometrically encodes the properties of orbital integrals, to the Hitchin fibration, which is global and whose geometric properties are easier to access. The mechanism that supplies the relation between the two is the product formula that Ngô established over the anisotropic locus $\mathcal{A}^{\mathrm{ani}}$ of the Hitchin base $\mathcal{A}$ in [70, théorème 4.6] and [71, proposition 4.15.1] and expected to also hold over the larger generically regular semisimple locus $\mathcal{A}^{\heartsuit} \subset \mathcal{A}$ in [71, before corollaire 4.15.2]. The product formula over $\mathcal{A}^{\heartsuit}$ has already been used, for instance, in [88, Proposition 2.4.1], [89,

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Section 5.5, equation (34)], or [72, proof of Proposition 6.6.3 (1)], and one of the main goals of this article is to establish it in Theorem 4.3.8.

Roughly speaking, the product formula is a geometric incarnation of the Beauville-Laszlo glueing for torsors: it translates this glueing into geometric properties of the morphism of algebraic stacks that relates affine Springer fibers, which parametrize torsors over formal disks $R[\![t]\!]$, to Hitchin fibers, which parametrize torsors over a fixed proper smooth curve X_R (for a variable base ring R). Under this dictionary, the product formula eventually reduces to a statement that torsors on X_R are obtained from the "Kostant-Hitchin torsor" over a fixed open $U_R \subset X_R$ by glueing along the punctured formal disks $R(\!(t)\!)$ at the R-points in $X \setminus U$. One is thus led to studying torsors over $R(\!(t)\!)$.

Over $\mathcal{A}^{\mathrm{ani}}$ the Hitchin fibration is separated and the intervening stacks are Deligne-Mumford. For the product formula, these additional properties allowed Ngô to reduce to only considering those R that are algebraically closed fields k, a case in which $k(\!(t)\!)$ is a field with relatively simple arithmetic. Over $\mathcal{A}^{\heartsuit}$, however, such a reduction does not seem available, and we need to study more general $R(\!(t)\!)$.

1.2. Torsors under tori over $R(\!(t)\!)$

The product formula says that the comparison morphism is a universal homeomorphism, so, due to the valuative criteria for stacks, the R that are most relevant are fields and discrete valuation rings. Nevertheless, the valuative criterion for universal closedness assumes that the map is quasi-compact, so, to avoid verifying this assumption directly, it is convenient to allow more general R (see Lemma 4.3.7). Our R will in fact be seminormal, strictly Henselian, and local, and the key torsor-theoretic input to the product formula is then Theorem 3.2.4: for such an R and an $R(\!(t)\!)$-torus T that splits over a finite étale Galois cover whose degree is invertible in R,

$$H^1(R(\!(t)\!), T) \cong 0. \tag{1.2.1}$$

Relative purity results from [3, exposé XVI], whose essential input is the relative Abhyankar's lemma, reduce this vanishing to $T = \mathbb{G}_m$. In this case, there is in fact a general formula

$$\mathrm{Pic}(R(\!(t)\!)) \cong \mathrm{Pic}(R[t^{-1}]) \oplus H^1_{\text{ét}}(R, \mathbb{Z}) \tag{1.2.2}$$

due to Gabber [35] that is valid for any ring R and is presented in the slightly more general setting of an arbitrary R-torus in Theorem 3.1.7. For seminormal R, we have $\mathrm{Pic}(R[t^{-1}]) \cong \mathrm{Pic}(R)$, so if R is also strictly Henselian local, then all the terms in (1.2.2) vanish and (1.2.1) follows.

In addition, the vanishing (1.2.1) implies that for a seminormal, strictly Henselian, local ring R and any $n > 0$ less than any positive residue characteristic of R, every regular semisimple $n \times n$ matrix with entries in $R(\!(t)\!)$ is conjugate to its companion matrix—see Theorem 4.2.14, which gives a general conjugacy to a Kostant section result of this type.

Overall the argument for the product formula is fairly short—it suffices to read §§3.1–3.2, §4.3, and review §4.2—but we decided to complement it with the following improvements and generalizations to various broadly useful results that enter into its proof.

1.3. Algebraization of torsors and approximation

A practical deficiency of the Laurent power series ring $R(\!(t)\!)$ is that its formation does not commute with filtered direct limits and quotients in R, so one often prefers its Henselian counterpart $R\{t\}[\frac{1}{t}]$ reviewed in §2.1.2. We show that such "algebraization" does not affect torsors: by Corollary 2.1.22, for any ring R and any smooth, quasi-affine, $R\{t\}[\frac{1}{t}]$-group G,

$$H^1(R\{t\}[\tfrac{1}{t}], G) \xrightarrow{\sim} H^1(R(\!(t)\!), G), \tag{1.3.1}$$

which generalizes a result of Gabber-Ramero [36, Theorem 5.8.14] valid in the presence of a suitable embedding $G \hookrightarrow \mathrm{GL}_n$. To prove (1.3.1), we exhibit a general procedure for showing that

$$F(R\{t\}[\tfrac{1}{t}]) \xrightarrow{\sim} F(R(\!(t)\!))$$

for functors F that are invariant under Henselian pairs: the idea, which appears to be due to Gabber, is to consider the ring of t-adic Cauchy sequences (and double sequences) valued in $R\{t\}[\frac{1}{t}]$ and to show that this ring is Henselian along the ideal of nil sequences, see Lemma 2.1.13 and Theorem 2.1.15. To verify that our functor $F(-) = H^1(-, G)$ is invariant under Henselian pairs, we use recent results on Tannaka duality for algebraic stacks, see Proposition 2.1.4 and Theorem 2.1.6.

The idea of considering Cauchy sequences and, more generally, Cauchy nets also leads to a new proof and a generalization of the Elkik approximation theorem, for which we exhibit new non-Noetherian versions in Theorems 2.2.2, 2.2.10 and 2.2.17. We then use them to extend the algebraization results to non-affine settings in §2.3: for instance, we show that for a Noetherian ring R that is Henselian along an ideal J and the J-adic completion $\widehat{R}$,

$$\mathrm{Br}(U) \xrightarrow{\sim} \mathrm{Br}(U_{\widehat{R}}) \quad \text{for every open} \quad \mathrm{Spec}(R) \setminus V(J) \subset U \subset \mathrm{Spec}(R),$$

a result that was announced in [32, Theorem 2.8 (i)]; see Corollary 2.3.5 for further statements of this sort and the results preceding it in §2.3 for sharper non-Noetherian versions. For a concrete situation in which such a passage to completion is useful, see [19, Proposition 3.3 and the proof of Theorem 5.3].

1.4. The Chevalley isomorphism and small characteristics

The construction of the Hitchin fibration for a reductive group G with Lie algebra $\mathfrak{g}$ rests on the Chevalley isomorphism

$$\mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W, \tag{1.4.1}$$

where G acts on $\mathfrak{g}$ by the adjoint action, $\mathfrak{t}$ is the Lie algebra of a maximal torus $T \subset G$, and $W := N_G(T)/T$ is the Weyl group. In Theorem 4.1.10, we give a short proof for (1.4.1) that is new even over $\mathbb{C}$ but works over any base scheme S as long as G is *root-smooth* (see §4.1.1; this condition holds if 2 is invertible on S or if the geometric fibers of G avoid types C_n). The main idea is to consider the Grothendieck alteration

$$\widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g},$$

where $\widetilde{\mathfrak{g}}$ is the Lie algebra of the universal Borel subgroup of G, and to extend the W-action from the regular semisimple locus $\widetilde{\mathfrak{g}}^{\mathrm{rs}}$ to the maximal locus $\widetilde{\mathfrak{g}}^{\mathrm{fin}} = \widetilde{\mathfrak{g}}^{\mathrm{reg}}$ over which the