ISSN 0012-9593

quatrième série - tome 55 fascicule 3 mai-juin 2022

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

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Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. Sainte-Claire Deville
de 1883 à 1888 par H. Debray
de 1889 à 1900 par C. Hermite
de 1901 à 1917 par G. Darboux
de 1918 à 1941 par É. Picard
de 1942 à 1967 par P. Montel

Comité de rédaction au 1er octobre 2021

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45, rue d'Ulm, 75230 Paris Cedex 05, France.
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Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
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Tarifs

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.

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ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4ᵉ série, t. 55, 2022, p. 865 à 903

DIMERS AND CIRCLE PATTERNS

BY Richard KENYON, Wai Yeung LAM, Sanjay RAMASSAMY
AND Marianna RUSSKIKH

ABSTRACT. – We establish a correspondence between the dimer model on a bipartite graph and a circle pattern with the combinatorics of that graph, which holds for graphs that are either planar or embedded on the torus. The set of positive face weights on the graph gives a set of global coordinates on the space of circle patterns with embedded dual. Under this correspondence, which extends the previously known isoradial case, the urban renewal (the local move for dimer models) is equivalent to the Miquel move (the local move for circle patterns). As a consequence, we show that *Miquel dynamics* on circle patterns is a discrete integrable system governed by the octahedron recurrence. As special cases of these circle pattern embeddings, we recover harmonic embeddings for resistor networks and s-embeddings for the Ising model.

RÉSUMÉ. – Nous établissons une correspondance entre le modèle de dimères sur un graphe bipartite et un agencement de cercles avec la combinatoire de ce graphe, valable pour des graphes plongés sur le plan ou sur le tore. Les poids positifs sur les faces du graphe fournissent des coordonnées globales sur l'espace des agencements de cercles dont le dual est plongé. Via cette correspondance, qui étend le cas isoradial découvert précédemment, le renouvellement urbain (mouvement local pour les modèles de dimères) est équivalent au mouvement de Miquel (mouvement local pour les agencements de cercles). Il en découle que la dynamique de Miquel sur les agencements de cercles est un système intégrable discret gouverné par la récurrence de l'octaèdre. Comme cas particuliers de ces plongements comme agencements de cercles, on retrouve les plongements harmoniques pour les réseaux de résistances et les s-plongements pour le modèle d'Ising.

1. Introduction

The *bipartite planar dimer model* is the study of random perfect matchings ("dimer coverings") of a bipartite planar graph. The dimer model is a classical statistical mechanics model, and can be analyzed using determinantal methods: partition functions and correlation kernels are computed by determinants of associated matrices defined from the weighted graph [22]. Several other two-dimensional models of statistical mechanics, including the Ising model and the spanning tree model, can be regarded as special cases of the dimer

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model by subdividing the underlying graph [13, 43, 11, 27]. Natural parameters for the dimer model, defining the underlying probability measure, are *face weights*, which are positive real parameters on the bounded faces of the graph [18].

A *circle pattern* is a realization of a graph in $\mathbb{C}$ with cyclic faces, i.e., where all vertices on a face lie on a circle. Circle patterns are central objects in discrete differential geometry, related to (hyperbolic) polyhedra, Teichmüller space, and discrete conformal geometry. For example, following original ideas of William Thurston, two circle patterns with the same intersection angles are considered discretely conformally equivalent, see e.g., [8].

In [26] a relation was found between a special subset of dimer models, called *critical dimer models*, and *isoradial circle patterns*, i.e., circle patterns in which all the circles have the same radius. The partition function and various probabilistic quantities were related to the underlying 3D hyperbolic geometry. At that time there was no clear relation between general dimer models and general circle patterns and this question was raised again in [7] and [40].

The main purpose of this paper is to answer this question, establishing a correspondence between face-weighted bipartite planar graphs and circle patterns, which generalizes the isoradial case. This correspondence is formulated for two classes of planar graphs, finite graphs and infinite bi-periodic graphs. Under this correspondence, dimer face weights correspond to biratios of distances between circle centers. A major feature of this correspondence is to identify the *spider move* (also known as *urban renewal*, or cluster mutation), which is a local move for the dimer model, to an application of Miquel's six-circles theorem to the underlying circle pattern. This establishes a new connection between circle patterns and cluster algebras.

The circle patterns arising under this correspondence are those with a bipartite graph and with an *embedded dual*, where the dual graph is the graph of circle centers. Having embedded dual does not imply that the primal pattern is embedded, although the set of circle patterns with embedded dual includes all embedded circle patterns in which each face contains its circumcenter. Centers of bipartite circle patterns arise in various places. They coincide with the crease patterns of origami that are locally flat-foldable [19]. In discrete differential geometry, they are called conical meshes [39, 36] and related to discrete minimal surfaces [31]. Circle center embeddings are also considered in [10] under the name of t-embeddings with an emphasis on the convergence of discrete holomorphic functions to continuous ones in the small mesh size limit, i.e., when the circle radii tend to 0.

This correspondence between dimer models of statistical mechanics and circle patterns from discrete differential geometry should allow to transfer results from one field to the other. As a first application of this correspondence, we show that *Miquel dynamics*, a discrete-time dynamical system for periodic circle patterns introduced in [40] and also studied in [17], is a discrete integrable system governed by the octahedron recurrence, answering a conjecture made in [40]. In the dimer model our natural parameters are the positive face weights, which correspond on the level of circle patterns to embedded circle centers. However our results apply to general real face weights and non-embedded circle centers as well; in particular Miquel dynamics is algebraic in nature and the signs of the weights do not matter.

A central question in 2D statistical mechanics is to find embeddings of planar graphs which are adapted to a model and at the same time universal, i.e., with a definition valid for any planar graph, see e.g., [5]. While the definition of a statistical mechanics model on a planar graph (e.g., random walk, dimer model, Ising model) does not depend on the embedding of the graph, stating and proving scaling limit results to conformally covariant objects such as Brownian motion or SLE curves requires one to pick an appropriate embedding for the graph. *Harmonic embeddings* (also known as Tutte embeddings) provide such adapted embeddings for resistor networks and random walks (see e.g., [6]). The *s-embeddings* recently introduced by Chelkak [9] (see also [32]) are embeddings adapted to the Ising model.

Our main result is that circle center embeddings are the right universal framework to study the planar bipartite dimer model. A first indication of this is the aforementioned compatibility between the local moves for the dimer model and for circle patterns. A second indication is that both resistor networks and the Ising model on planar graphs can be seen as special cases of the bipartite dimer model [27, 11] and we show in this article that both harmonic embeddings and s-embeddings arise as special cases of circle center embeddings.

There is an intriguing algebraic similarity between the dimer model and Teichmüller theory: The face weights describing the dimer model [18] and the shear coordinates for Teichmüller space [14] both behave like X-variables from cluster algebras. The relation between Teichmüller theory and circle patterns together with the correspondence between dimers and circle patterns could help to shed light on this similarity.

During the completion of this work, a preprint by Affolter [2] appeared, which shows how to go from circle patterns to dimers and observes that the Miquel move is governed by the central relation. Affolter notes that there is some information missing to recover the circle pattern from the X variables. We provide here a complete picture, both in the planar and torus cases.

Organization of the paper

In Section 3, we introduce circle center embeddings associated with bipartite graphs with positive face weights in the planar case. Section 4 is devoted to circle center embeddings in the torus case. In Section 5 we show the equivalence between the spider move/urban renewal for the bipartite dimer model and the central move coming from Miquel's theorem for circle patterns. In particular this gives a cluster algebra structure underlying Miquel dynamics. Section 6 is devoted to translating into circle geometry the generalized Temperley bijection between resistor networks and dimer models. In Section 7 we show that the s-embeddings for the Ising model arise as a special case of circle center embeddings.

2. Background on dimers and the Kasteleyn matrix

For general background on the dimer model, see [23]. A *dimer cover*, or *perfect matching*, of a graph $\mathcal{G}$ is a set of edges with the property that every vertex is contained in exactly one edge of the set. We assume our graphs are finite, connected and embeddable either on the plane or on the torus. A graph is *nondegenerate* (for the dimer model) if it has dimer covers, and each edge occurs in some dimer cover.

If $\nu : E(\mathcal{G}) \to \mathbb{R}_{>0}$ is a positive weight function on edges of $\mathcal{G}$, we associate a weight $\nu(m) = \prod_{e\in m} \nu(e)$ to a dimer cover which is the product of its edge weights. We can also associate to this data a probability measure μ on the set M of dimer covers, giving