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MACROSCOPIC SCALAR CURVATURE AND LOCAL COLLAPSING

BY Stéphane SABOURAU

ABSTRACT. – Consider a closed n-manifold M admitting a negatively curved Riemannian metric. We show that for every Riemannian metric on M of sufficiently small volume, there is a point in the universal cover of M such that the volume of every ball of radius $r \geq 1$ centered at this point is greater or equal to the volume of the ball of the same radius in the hyperbolic n-space. We also give an interpretation of this result in terms of macroscopic scalar curvature. This result, which holds more generally in the context of polyhedral length spaces, is related to a question of Guth. Its proof relies on a generalization of recent progress in metric geometry about the Alexandrov/Urysohn width involving the volume of balls of radius in a certain range with collapsing at different scales.

RÉSUMÉ. – Considérons une n-variété fermée M admettant une métrique riemannienne à courbure strictement négative. Nous montrons que pour toute métrique riemannienne sur M de volume suffisamment petit, il existe un point dans le revêtement universel de M tel que le volume des boules de rayon $r \geq 1$ centrées en ce point est supérieur ou égal au volume de la boule de même rayon dans l'espace hyperbolique de dimension n. Nous donnons également une interprétation de ce résultat en termes de courbure scalaire macroscopique. Ce résultat, valable plus généralement dans le contexte des espaces de longueur polyédraux, est lié à une question de Guth. Sa démonstration repose sur une généralisation de progrès récents en géométrie métrique concernant la largeur d'Alexandrov/Urysohn mettant en jeu le volume des boules de rayon d'une certaine amplitude avec un effrondrement à différentes échelles.

1. Introduction

The scalar curvature of a closed Riemannian n-manifold M describes how the volume of infinitesimal balls in M compares to the volume of infinitesimal balls in the Euclidean n-space. More precisely, the volume expansion of a ball of radius r centered at $x \in M$ satisfies

$$\text{vol}(B(x,r)) = \omega_n \, r^n \left(1 - \frac{\text{scal}(M,x)}{6(n+2)} \, r^2 + O(r^3)\right) \tag{1.1}$$

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as r goes to zero, where $\mathrm{scal}(M,x)$ is the scalar curvature of M at x and ω_n is the volume of a unit ball in the Euclidean n-space; see [10, Theorem 3.98]. Understanding the relationship between scalar curvature and the topology of M is a major problem in Riemannian geometry.

In this article, we will be interested in macroscopic versions of the following conjecture attributed to Schoen (which follows from a conjecture of Schoen about the Yamabe invariant of hyperbolic manifolds); see [26] and [16]. This conjecture was also stated by Gromov [12, 3.A] with nonsharp constants.

CONJECTURE 1.1 (Schoen). – Let (M, hyp) be a closed hyperbolic n-manifold and let g be another Riemannian metric on M. If $\mathrm{scal}(g,x) \geq \mathrm{scal}(\mathrm{hyp})$ for every $x \in M$ then

$$\mathrm{vol}(M,g) \geq \mathrm{vol}(M,\mathrm{hyp}).$$

Equivalently, using (1.1), if $\mathrm{vol}(M,g) < \mathrm{vol}(M,\mathrm{hyp})$ then there exists $x_0 \in M$ such that

$$\mathrm{vol}_g(B(x_0,r)) > \mathrm{vol}_{\mathrm{hyp}}(B(r))$$

for every $r > 0$ small enough.

This conjecture is true in dimension 2 by the Gauss-Bonnet formula and in dimension 3 from Perelman's work; see [21, Proposition 93.10]. In higher dimension, it also holds true for Riemannian metrics close enough to the hyperbolic one, see [5, Corollaire C], or if one replaces scalar curvature with Ricci curvature, see [6].

Following [14], this leads us to introduce the following notion. The *macroscopic scalar curvature* of a closed Riemannian n-manifold M at scale r at $x \in M$, denoted by $\mathrm{scal}_r(M,x)$, is defined as the unique real s such that

$$\mathrm{vol}(B_{\tilde{M}}(\tilde{x},r)) = \mathrm{vol}(B_{\mathbb{H}^n_s}(r)),$$

where $\tilde{x}$ is a lift of x in the universal cover $\tilde{M}$ of M and $\mathbb{H}^n_s$ is the simply-connected n-dimensional space form with constant curvature s. It is more conveniently characterized as follows

$$\mathrm{scal}_r(M,x) \leq s \text{ if and only if } \mathrm{vol}(B_{\tilde{M}}(\tilde{x},r)) \geq \mathrm{vol}(B_{\mathbb{H}^n_s}(r)).$$

For example, the macroscopic scalar curvature of a flat torus at any scale is zero. Note that this property fails if one does not take balls in the universal cover of M, but only in M, in the definition of the macroscopic scalar curvature, as otherwise, it would be positive at a large enough scale. By (1.1), at infinitesimally small scale, we have

$$\lim_{r\to 0} \mathrm{scal}_r(M,x) = \mathrm{scal}(M,x).$$

In a different direction, the macroscopic scalar curvature at large enough scale provides information on the exponential growth rate of the volume of balls in the universal cover of M, also known as the volume entropy, a much-studied geometric invariant related to the growth of the fundamental group and the dynamics of the geodesic flow. This leads us to define

$$V_{\tilde{M}}(r) = \sup_{\tilde{x}\in\tilde{M}} \mathrm{vol}(B(\tilde{x},r))$$

as the maximal volume of a ball of radius r in the universal cover of M. As explained in [16] and [14], the celebrated theorem of Besson, Courtois and Gallot [6] on the minimal volume entropy provides a macroscopic version of Schoen's Conjecture 1.1 at large enough scales.

Stated in a way suited for comparison (ignoring its rigidity counterpart), this result takes the following form.

THEOREM 1.2 (Besson-Courtois-Gallot [6]). – *Let* (M, hyp) *be a closed hyperbolic* n*-manifold and let* g *be another Riemannian metric on* M*. If* $\mathrm{vol}(M, g) < \mathrm{vol}(M, \mathrm{hyp})$ *then there exists* $r_0 > 0$ *(depending on* g*) such that for every* $r \geq r_0$

$$V_{\tilde{M}}(r) > V_{\mathbb{H}^n}(r).$$

In particular, if $\mathrm{scal}_r(g, x) > \mathrm{scal}_r(\mathrm{hyp})$ *for every* r *large enough and every* $x \in M$ *then* $\mathrm{vol}(M, g) > \mathrm{vol}(M, \mathrm{hyp})$.

A version of this theorem was first established by A. Katok [20] in dimension 2 and a nonsharp version was obtained before by Gromov [11] in every dimension.

In [16], Guth asks for an estimate on r_0 after proving the following nonsharp macroscopic version of Schoen's conjecture.

THEOREM 1.3 (Guth [16]). – *Let* (M, hyp) *be a closed hyperbolic* n*-manifold and let* g *be another Riemannian metric on* M*. Then, for every* $r \geq 1$*, there exists a constant* $\delta_{n,r} > 0$ *depending only on* n *and* r*, such that if* $\mathrm{vol}(M, g) \leq \delta_{n,r}\, \mathrm{vol}(M, \mathrm{hyp})$ *then*

$$V_{\tilde{M}}(r) \geq V_{\mathbb{H}^n}(r).$$

In other words, if $\mathrm{scal}_r(g, x) \geq \mathrm{scal}_r(\mathrm{hyp})$ *for every* $x \in M$ *then* $\mathrm{vol}(M, g) \geq \delta_{n,r}\, \mathrm{vol}(M, \mathrm{hyp})$.

Further volume lower bounds have recently been obtained by Alpert and Funano for hypersurfaces in closed manifolds with macroscopic scalar curvature bounded below as a consequence of this result; see [2] and [1].

Theorem 1.3 gives relatively better volume estimates for unit balls (that is, for $r = 1$) than for balls of large radius as the constant $\delta_{n,r}$ falls off exponentially or faster with r. In [16], Guth suggests that one could try to combine the approaches of [11], [6] and [16] to obtain a uniform volume estimate with $\delta_{n,r} = \delta_n$ depending only on n. Such a uniform bound was obtained for surfaces by Karam [19]. In higher dimension, Balacheff and Karam [4] proved a similar result for negatively curved metrics g using techniques developed in [11].

In this article, we establish the following result in this direction, following a different approach. See Theorem 3.7, Corollary 3.9 and Corollary 3.10 for more general statements.

THEOREM 1.4. – *Let* (M, hyp) *be a closed hyperbolic* n*-manifold and let* g *be another Riemannian metric on* M*. Then, there exists a constant* $\delta'_n > 0$ *depending only on* n*, such that if* $\mathrm{vol}(M, g) \leq \delta'_n$ *then*

$$V_{\tilde{M}}(r) \geq V_{\mathbb{H}^n}(r)$$

for every $r \geq 1$.

In other words, if $\mathrm{scal}_r(g, x) \geq \mathrm{scal}_r(\mathrm{hyp})$ *for some* $r \geq 1$ *and every* $x \in M$ *then* $\mathrm{vol}(M, g) \geq \delta'_n$.