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Antoine JULIA

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
Email : annales@ens.fr

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64. Fax : (33) 04 91 41 17 51
Email : abonnements@smf.emath.fr

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A GENERALIZED STOKES' THEOREM ON INTEGRAL CURRENTS

BY ANTOINE JULIA

ABSTRACT. – The purpose of this paper is to study the validity of Stokes' Theorem on singular submanifolds, for differential forms with singularities in Euclidean space. The results are presented in the context of Lebesgue Integration, but their proofs involve techniques from gauge integration in the spirit of R. Henstock, J. Kurzweil and W. F. Pfeffer. We prove a generalized Stokes' Theorem on integral currents of dimension m whose singular sets have finite $m-1$ dimensional intrinsic Minkowski content. This condition applies in particular to codimension 1 mass minimizing integral currents with smooth boundary and to semi-algebraic chains. Conversely, we give an example of integral current of dimension 2 in $\mathbb{R}^3$, with only one singular point, to which our version of Stokes' Theorem does not apply.

RÉSUMÉ. – On étudie la validité du théorème de Stokes sur des sous-variétés singulières de l'espace euclidien pour des formes différentielles admettant des singularités. Les résultats sont présentés dans le contexte de l'intégrale de Lebesgue, mais sont démontrés grâce à des techniques d'intégration non-absolument convergente dans l'esprit de W. F. Pfeffer, et de l'intégrale de Henstock-Kurzweil. On démontre un théorème de Stokes généralisé sur les courants entiers de dimension m dont les ensembles singuliers ont un contenu de Minkowski relatif de dimension $m-1$ fini. De tels courants incluent les courants entiers minimiseurs de masse et les chaînes semi-algébriques. Par contraste, on construit un courant entier de dimension 2 dans $\mathbb{R}^3$ ayant un seul point singulier et ne vérifiant pas ce théorème de Stokes généralisé.

1. Introduction and main results

Stokes' Theorem is a key result in geometry and analysis. From the analytic point of view, it can be seen as a general version of the Fundamental Theorem of Calculus and of the Divergence Theorem. It lies at the core of integration by parts and thereby of the notion of

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weak solution of partial differential equations. On the geometric side, Stokes' Theorem is key to de Rham cohomology.

In geometric analysis, the notion of calibration (see for instance [30, 18]) connects these two points of view. As for many variational problems one expects minimizers to exhibit singularities, a corresponding calibration is then expected to be singular as well. It is natural to try and generalize Stokes' Theorem to singular forms and varieties. Another motivation is the study of PDEs on singular surfaces.

The classical Stokes' Theorem is usually stated as follows:

THEOREM (Stokes' Theorem). – *If M is a compact m-dimensional oriented C^1 submanifold of $\mathbb{R}^n$, with boundary ∂M and ω is a C^1 differential form of degree $m-1$ on M, then there holds*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega. \tag{1}$$

The main question behind the present work is the following:

QUESTION. – *Suppose now that M is a singular submanifold with singular set E_M and that ω has singularities in the set E_ω. Under what conditions on the sets E_M and E_ω, and possibly on the types of singularities, does identity* (1) *still hold?*

Let us first review some classical answers to these questions. If $m = n = 1$, M is a compact interval and $\omega = f$ is a function, this amounts to proving a generalized Fundamental Theorem of Calculus. It is known that if f is continuous, differentiable except in a countable set and f' is Lebesgue integrable, then the primitive of f' is equal to f up to a constant. However, the Lebesgue integrability condition is not automatically satisfied, even if f is differentiable everywhere—consider for example the continous extension to $[-\pi^{-1/2}, \pi^{-1/2}]$ of the function $x \mapsto x^2 \sin(x^{-2})$.

This problem was solved by introducing a new type of integral on intervals, whose first formulations were given by A. Denjoy and O. Perron. It is now known as the Henstock-Kurzweil integral [23, 19], and can be constructed in a way very similar to the Riemann Integral, though it is more general than the Lebesgue Integral. The books [39, 12, 29] contain detailed presentations of these questions. The main advantage of the formulation using Riemann Sums lies in the focus on the domain of integration—as opposed to the focus on the range, as in the Lebesgue Integral. Indeed focusing on the domain provides a better control on the behavior of the function near pointwise singularities, by the mean of a *gauge*, i.e., a non-negative function defined on the domain and controlling the size of the elements in the Riemann sums. In particular, this type of method can also yield results for the Lebesgue integral.

If $m = n \geqslant 1$ and M represents a bounded set of finite perimeter, a method of gauge integration was developped by W. F. Pfeffer [35], following in particular works of J. Mařík [27], J. Mawhin [28]. The main result of Pfeffer Integration is a generalized Divergence Theorem, which we rephrase below as Theorem 0 using the notation of this paper. W. F. Pfeffer's result extends the celebrated Theorem of E. De Giorgi and H. Federer (see [8, Theorem 4.5.6]), which states that if $A \subset \mathbb{R}^m$ is a bounded set of finite perimeter and $\mathbf{v}$ is a Lipschitz vector field, then the Divergence Theorem holds:

$$\int_A \operatorname{div} \mathbf{v} = \int_{\partial_* A} \mathbf{v} \cdot \nu_A \, \mathrm{d}\mathscr{H}^{m-1},$$

where $\partial_* A$ is the reduced boundary of A and ν_A is its outer normal and for $s \in [0, n]$, $\mathscr{H}^s$ denotes the Hausdorff measure of dimension s.

We now turn to the case where $m \leqslant n$. If M is a C^1 submanifold, this reduces to the flat case ($m = n$) by C^1 triangulation and changes of variables. In order to study singular submanifolds, we choose to work in the setting of integral currents in Euclidean space. These currents were introduced in [10] by H. Federer and W. H. Fleming, who presented them as a satisfactory class of "k dimensional domain of integration in Euclidean n-space". We will mostly follow the notation from the classical book [8], let us introduce some of it now (see also Section 2).

Integral currents form a subset of the currents in the sense of de Rham: A current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is a continuous linear operator on $\mathcal{D}^m(\mathbb{R}^n)$, the space of smooth differential forms of degree m in $\mathbb{R}^n$ with compact support. The *boundary* of T is the current $\partial T \in \mathcal{D}_{m-1}(\mathbb{R}^n)$ defined for $\omega \in \mathcal{D}^{m-1}(\mathbb{R}^n)$ by $\partial T(\omega) = T(\mathrm{d}\omega)$. The *mass* of a current T of dimension m is defined as

$$\mathbf{M}(T) = \sup\{T(\omega),\ \omega \in \mathcal{D}^m(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n,\ |\omega(x)| < 1\}.$$

In particular, if T represents the oriented submanifold M, i.e., if $T(\omega) = \int_M \omega$ for $\omega \in \mathcal{D}^m(\mathbb{R}^n)$, then there holds $\mathbf{M}(T) = \mathscr{H}^m(M)$.

For compactness purposes, one prefers to work with Lipschitz instead of C^1 maps. This leads to the notion of rectifiability: a set is *m-rectifiable*, if it can be covered up to an $\mathscr{H}^m$ null set by a countable union of Lipschitz images of $\mathbb{R}^m$. We say that $T \in \mathcal{D}_m(\mathbb{R}^n)$ is a *rectifiable current* if it has compact support (denoted by $\operatorname{spt} T$) and can be represented by a triple $(M, \theta, \overrightarrow{T})$, where

(a) $M \subset \mathbb{R}^n$ is m-rectifiable,

(b) $\theta : M \to \mathbb{Z}$ is integrable with respect to $\mathscr{H}^m \llcorner M$,,

(c) $\overrightarrow{T}$ is an $\mathscr{H}^m$-measurable unit m-vector field, $\mathscr{H}^m \llcorner M$, a.-e. tangent to M.

We then write $T = \|T\| \wedge \overrightarrow{T}$, where the Radon measure $\|T\| = \theta \mathscr{H}^m \llcorner M$ is called the *carrying measure* of T; there holds $\mathbf{M}(T) = \|T\|(\mathbb{R}^n)$. The action of T on $\omega \in \mathcal{D}^m(\mathbb{R}^n)$ is then

$$T(\omega) = \int_M \langle \omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|T\|(x),$$

where $\langle \cdot, \cdot \rangle : \Lambda^m(\mathbb{R}^n) \times \Lambda_m(\mathbb{R}^n) \to \mathbb{R}$ represents the duality pairing between m-covectors and m-vectors in $\mathbb{R}^n$. Finally, a current $T \in \mathcal{D}_m(\mathbb{R}^n)$ is an *integral current* ($T \in \mathbf{I}_m(\mathbb{R}^n)$) if both T and ∂T are rectifiable currents.

Given $T \in \mathbf{I}_m(\mathbb{R}^n)$ and a smooth form $\omega \in \mathcal{D}^m(\mathbb{R}^n)$, the identity $T(\mathrm{d}\omega) = \partial T(\omega)$ can be written as

$$\int \langle \mathrm{d}\omega(x), \overrightarrow{T}(x) \rangle \, \mathrm{d}\|T\|(x) = \int \langle \omega(x), \overrightarrow{\partial T}(x) \rangle \, \mathrm{d}\|\partial T\|(x). \tag{2}$$