ISSN 0012-9593

quatrième série - tome 55 fascicule 4 juillet-août 2022

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Édition et abonnements / *Publication and subscriptions*

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Tarifs

Abonnement électronique : 441 euros.
Abonnement avec supplément papier :
Europe : 619 €. Hors Europe : 698 € ($ 985). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Fabien Durand
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 55, 2022, p. 969 à 1002

A DICHOTOMY FOR MEASURES OF MAXIMAL ENTROPY NEAR TIME-ONE MAPS OF TRANSITIVE ANOSOV FLOWS

BY Jérôme BUZZI, Todd FISHER AND Ali TAHZIBI

ABSTRACT. – We show that time-one maps of transitive Anosov flows of compact manifolds are accumulated by diffeomorphisms robustly satisfying the following dichotomy: either all of the measures of maximal entropy are non-hyperbolic, or there are exactly two ergodic measures of maximal entropy, one with a positive central exponent and the other with a negative central exponent.

We establish this dichotomy for certain partially hyperbolic diffeomorphisms isotopic to the identity whenever both of their strong foliations are minimal. Our proof builds on the approach developed by Margulis for Anosov flows where he constructs suitable families of measures on the dynamical foliations.

RÉSUMÉ. – Nous montrons que l'application du temps 1 de tout flot d'Anosov transitif est dans l'adhérence des difféomorphismes présentant robustement la dichotomie suivante: ou bien aucune mesure maximisant l'entropie n'est hyperbolique, ou bien il existe exactement deux mesures ergodiques maximisant l'entropie, l'une ayant un exposant central strictement positif, l'autre strictement négatif.

Nous établissons cette dichotomie pour certains difféomorphismes partiellement hyperboliques isotopes à l'identité sous l'hypothèse de la minimalité de leurs feuilletages invariants forts. Notre preuve s'appuie sur l'approche développée par Margulis dans le cas des flots d'Anosov et la construction de familles de mesures convenables sur les différents feuilletages dynamiques.

1. Introduction

In his pioneering work [32], Margulis studied measures of maximal entropy of geodesic flows in order to count closed geodesics for manifolds with variable negative curvature. More precisely, he constructed a family of measures $\{m_x\}_{x\in M}$ such that for all $x \in M$ the measure m_x is carried by the unstable manifold at x, and for all $t \in \mathbb{R}$ we have

$$(\varphi^t)_* m_x = e^{-t\cdot h_{\mathrm{top}}(\varphi)} m_{\varphi^t x}.$$

T.F. is supported by Simons Foundation grant # 239708. A.T is supported by FAPESP 107/06463-3 and CNPq (PQ) 303025/2015-8. A.T.'s stay at the Institut de Mathématique d'Orsay was partially supported by the ANR project ISDEEC # ANR-16-CE40-0013.

 doi:10.24033/asens.2511

He then built an invariant probability measure which was observed to be a measure of maximal entropy and is now called the Bowen-Margulis measure. It was then proved to be the unique measure of maximal entropy. We refer to Ledrappier [28] for an introduction.

In this paper, we will extend Margulis' construction to a class of partially hyperbolic maps and obtain a striking dichotomy.

Theorem 1.1. – *If φ^t is a transitive Anosov flow on a compact manifold M, then there is an open set $\mathcal{U}$ in $\mathrm{Diff}^1(M)$ which contains φ^1 in its closure such that for any $f \in \mathcal{U} \cap \mathrm{Diff}^2(M)$ we have the following dichotomy:*

1. *either all the measures of maximal entropy have zero central Lyapunov exponents, or*
2. *there are exactly two ergodic measures of maximal entropy where one has a positive central exponent and the other has a negative central exponent, and both measures are Bernoulli.*

Related results

These results are part of a larger program to understand properties of entropy beyond uniform hyperbolicity. In that classical setting, say for a transitive Anosov diffeomorphism, there is a unique measure of maximal entropy. (1) Beyond the hyperbolic setting, even though there are a number of significant results [35, 11, 12] there are still many fundamental open questions. Partially hyperbolic diffeomorphisms with one-dimensional center have been studied as the "next nontrivial class". A MME always exists in this setting by entropy expansivity (see [15, 18, 31]). Its uniqueness is a more delicate question.

This uniqueness has been shown for certain systems that are *derived from Anosov*, a subclass introduced by Mañé, first for specific constructions, then in greater and greater generality [13, 16, 49, 20, 10].

The partially hyperbolic diffeomorphisms the center foliation of which is given by circles form another subclass with a more subtle behavior. Assuming accessibility, [39] has established the following dichotomy:

– either the dynamics is isometric in the center direction and there exists a unique MME which is nonhyperbolic; or
– there are multiple hyperbolic MMEs.

Strategy of proof

We introduce a new subclass of partially hyperbolic diffeomorphisms with one-dimensional center which we call *flow-type*. They are isotopic to the identity and the fundamental examples are the perturbations of time-one maps of Anosov flows. Our main result is Theorem 3.10: the above dichotomy holds for partially hyperbolic flow-type diffeomorphisms whose strong foliations are both minimal.

The uniqueness of the MME for a given sign of the central exponent (say nonpositive) follows from a variant of Margulis' approach. Namely, we first build a family of measures on the center-unstable leaves which is invariant under stable holonomies and projectively

(1) Throughout this paper, we will abbreviate *ergodic Borel probability measure maximizing the entropy* to MME. Recall that a measure maximizes the entropy if and only if it is invariant and almost all its ergodic components are MMEs.

invariant under the dynamics. Then we construct measures on unstable leaves, which we call Margulis u-conditionals. This is more difficult for maps than for flows.

We then use the entropy with respect to the unstable foliation as introduced by Ledrappier and Young [29] and an argument of Ledrappier [26] to show that, when its central exponent is nonpositive, a measure maximizes the entropy if and only if its disintegration along the unstable leaves is given by the Margulis conditionals.

A Hopf argument shows that if there is a MME with negative central exponent, then any MME with nonpositive central exponent must coincide with it. The symmetry between positive and negative central exponents in the hyperbolic case follows from the one-dimensionality of the central leaves: we associate to any measure with, say negative central exponent, an isomorphic one with nonnegative central exponent.

A hyperbolic MME is isomorphic to a Bernoulli shift times a circular permutation, according to a general result by Ben Ovadia [36]. The triviality of the permutation follows by considering iterates. This concludes the outline of the proof of Theorem 3.10.

Finally, to prove Theorem 1.1 we establish Theorem 3.11, i.e., we find open sets of partially hyperbolic flow-type diffeomorphisms with both strong foliations minimal near any time-one map of a transitive Anosov flow. We first show that such time-one maps are robustly flow-type, partially hyperbolic diffeomorphisms. Then Bonatti and Díaz [6] provide a perturbation ensuring robust transitivity. Lastly, by a further perturbation following Bonatti, Díaz, and Urès [7] we ensure robust minimality of both strong foliations. Theorem 1.1 now follows from Theorem 3.10.

The use of Margulis conditionals

The construction of Margulis has given rise to a large body of work, mainly devoted to the estimation of the number of periodic orbits, sometimes beyond the uniformly hyperbolic setting [25]. We refer to Sharp's survey in [33], the long awaited publication of Margulis' thesis. The works of Hamenstädt [22] and Hasselblatt [23] that give a geometric description of the Margulis conditionals $\{m_x\}_{x\in M}$ are perhaps closer to our concerns. We also note Plante's point of view [38] (see also [45, 43]): measures invariant under stable holonomies can be seen as transverse measures. We have not pursued this point of view here.

While this work was being written, we learned that a different but related approach has been developed in [14]. This approach can deal with equilibrium measures (i.e., generalizations of measures of maximal entropy taking into account a weight function) but requires non-expansion along the center. Separately, Jiagang Yang has told us that he also has some related results.

Comments

First, we note that our result, Theorem 3.10, will be proved for a class of partially hyperbolic diffeomorphisms which we call *flow-type* (see Def. 3.1). This class turns out