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HEEGNER POINTS, STARK-HEEGNER POINTS, AND DIAGONAL CLASSES

Massimo BERTOLINI, Henri DARMON, Victor ROTGER, Marco Adamo SEVESO & Rodolfo VENERUCCI

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HEEGNER POINTS, STARK-HEEGNER POINTS, AND DIAGONAL CLASSES

Massimo Bertolini, Henri Darmon, Victor Rotger,
Marco Adamo Seveso and Rodolfo Venerucci

Abstract.—This volume comprises four interrelated articles whose unifying theme is the study of Heegner and Stark-Heegner points, and their connections with the p-adic logarithm of certain global cohomology classes attached to a pair of weight one theta series of a common (imaginary or real) quadratic field. These global classes are obtained from p-adic deformations of diagonal classes attached to triples of modular forms of weight > 1, and naturally generalise a construction of Kato which one recovers when the two theta series are replaced by Eisenstein series of weight one. Understanding the extent to which such classes obtained via the p-adic interpolation of motivic cohomology classes are themselves motivic is a key motivation for this study. A second is the desire to show that Stark-Heegner points, whose global nature is still poorly understood theoretically, arise from classes in global Galois cohomology.

***Résumé* (Points de Heegner, points de Stark-Heegner et classes diagonales).**— Ce volume est constitué de 4 articles interdépendants dont le théme unificateur est l'étude des points de Heegner et de Stark-Heegner, et leurs relation avec certaines classes de cohomologie Galoisienne globales associées à une paire de séries theta de poids un du même corps quadratique (imaginaire ou réel). Ces classes proviennent de déformations p-adiques des classes diagonales associés à des triplets de formes modulaires de poids > 1, et généralisent une construction de Kato que l'on récupère quand les deux séries theta sont remplacés par des séries d'Eisenstein de poids un. Une des motivations pour cette étude est de comprendre dans quelle mesure de telles classes, obtenues par interpolation p-adique à partir de familles de classes motiviques, restent elles-mêmes motiviques. Ces résultats permettent aussi de démontrer que les points de Stark-Heegner, dont les propriétés d'algébricité sont encore complétement conjecturales, proviennent tout au moins de classes de cohomologie globales.

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RÉSUMÉS DES ARTICLES

Points de Stark-Heegner et classes diagonales
Henri Darmon & Victor Rotger 1

Les points de Stark-Heegner généralisent les points de Heegner quand un corps quadratique réel K remplace le corps quadratique imaginaire de la théorie de la multiplication complexe. La démarche p-adique qui les sous-tend fait que ces points sur une courbe elliptique E sont à priori locaux, définis sur une extension finie de $\mathbf{Q}_p$. On conjecture qu'ils sont de nature globale, qu'ils appartiennent aux groupes de Mordell-Weil de E sur certains corps de classe de K, et qu'ils satisfont une loi de réciprocité de Shimura décrivant l'action de $G_K := \mathrm{Gal}(\bar{K}/K)$. Les conjectures de [**15**] prédisent ainsi qu'une combinaison linéaire de points de Stark-Heegner pondérée par les valeurs d'un caractère ψ de G_K appartient au sous-espace propre correspondant du groupe Mordell-Weil de E sur le corps de classe découpé par ψ, et qu'elle est non triviale si et seulement si $L'(E/K, \psi, 1) \neq 0$. On démontre que cette combinaison linéaire provient tout au moins d'une classe globale dans la ψ-partie du pro-p groupe de Selmer de E, et qu'elle est non-triviale lorsque la dérivée première d'une certaine fonction L p-adique associée à E ne s'annule pas en ψ.

Familles p-adiques de cycles diagonaux
Henri Darmon & Victor Rotger 29

On construit une famille à trois variables de classes de cohomologie associée à des cycles diagonaux sur le produit triple de tours de courbes modulaires, et on démontre une loi de réciprocité qui réalise la fonction L p-adique d'un triplet de familles de Hida comme l'image de cette famille de classes de cohomologie par le régulateur Λ-adique de Perrin-Riou.

Lois de réciprocité pour les classes diagonales équilibrées
Massimo Bertolini, Marco Adamo Seveso & Rodolfo Venerucci 77

Cet article construit une classe diagonale $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ à trois variables dans la cohomologie de la représentation galoisienne associée à un triplet auto-dual $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ de familles de Hida. Le premier résultat principal (Théorème A de la Section 1.1) fournit une loi de réciprocité explicite reliant $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ à la fonction L p-adique

de Garrett-Rankin attachée à $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. La classe $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ s'obtient par interpolation p-adique des classes diagonales dans les groupes de Selmer à la Bloch-Kato des spécialisations de $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ aux triplets de poids classiques «équilibrés». On en déduit que la valeur de $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ en une spécialisation (f, g, h) de poids déséquilibré est une limite p-adique de classes cristallines. Le deuxième résultat principal (Théorème B de la Section 1.2) montre que l'obstruction à ce qu'une dérivée appropriée de $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ en (f, g, h) soit cristalline est contrôlée par la valeur centrale critique de la fonction L complexe de $f \otimes g \otimes h$.

Classes diagonales équilibrées et points rationaux sur les courbes elliptiques
MASSIMO BERTOLINI, MARCO ADAMO SEVESO & RODOLFO VENERUCCI 175

Soit A une courbe elliptique sur le corps des rationnels ayant réduction multiplicative en un premier p, et soit K un corps quadratique dans lequel p est inerte. Sous une hypothèse de Heegner généralisée (cf. article précédent) associe aux données (A, p, K) une classe diagonale dans le groupe de Selmer du module de Tate p-adique de A sur certains corps de classes d'anneau de K. Ces classes diagonales sont des limites p-adiques de classes de provenance géométrique, appartenant à la cohomologie de certains produits de variétés de Kuga-Sato. Le résultat principal de cet article relie ces classes diagonales aux logarithmes p-adiques de points de Heegner quand K est complexe, et de Stark-Heegner quand K est réel.

ABSTRACTS

Stark-Heegner points and diagonal classes
Henri Darmon & Victor Rotger 1

Stark-Heegner points are conjectural substitutes for Heegner points when the imaginary quadratic field of the theory of complex multiplication is replaced by a real quadratic field K. They are constructed analytically as local points on elliptic curves with multiplicative reduction at a prime p that remains inert in K, but are conjectured to be rational over ring class fields of K and to satisfy a Shimura reciprocity law describing the action of G_K on them. The main conjectures of [**15**] predict that any linear combination of Stark-Heegner points weighted by the values of a ring class character ψ of K should belong to the corresponding piece of the Mordell-Weil group over the associated ring class field, and should be non-trivial when $L'(E/K, \psi, 1) \neq 0$. Building on the results on families of diagonal classes described in the remaining contributions to this volume, this note explains how such linear combinations arise from global classes in the idoneous pro-p Selmer group, and are non-trivial when the first derivative of a weight-variable p-adic L-function $(/K, \psi)$ does not vanish at the point associated to $(E/K, \psi)$.

p-adic families of diagonal cycles
Henri Darmon & Victor Rotger 29

This note provides the construction of a three-variable family of cohomology classes arising from diagonal cycles on a triple product of towers of modular curves, and proves a reciprocity law relating it to the three variable triple-product p-adic L-function associated to a triple of Hida families by means of Perrin-Riou's Λ-adic regulator.

Reciprocity laws for balanced diagonal classes
Massimo Bertolini, Marco Adamo Seveso & Rodolfo Venerucci 77

This article constructs a 3-variable *balanced* diagonal class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in the cohomology of the Galois representation associated to a self-dual triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of p-adic Hida families. Its first main result (Theorem A of Section 1.1) establishes an explicit reciprocity law relating $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ to the *unbalanced* Garrett-Rankin p-adic L-function attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. The class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ arises from the p-adic

interpolation of diagonal classes in the Bloch-Kato Selmer groups of the specializations of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at balanced triples of classical weights. As a consequence, the value of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at a specialization (f, g, h) of $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at an unbalanced triple of classical weights is a p-adic limit of crystalline classes. Our second main result (Theorem B of Section 1.2) shows that the obstruction to the crystallinity of an appropriate derivative of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at (f, g, h) is encoded in the central critical value of the complex L-function of $f \otimes g \otimes h$.

Balanced diagonal classes and rational points on elliptic curves
MASSIMO BERTOLINI, MARCO ADAMO SEVESO & RODOLFO VENERUCCI 175

Let A be an elliptic curve over the rationals with multiplicative reduction at a prime p, and let K be a quadratic field in which p is inert. Under a generalized Heegner assumption, our previous contribution (see previous article) to this volume attaches to (A, p, K) balanced diagonal classes in the Selmer groups of the p-adic Tate module of A over certain ring class fields of K. These classes are obtained as p-adic limits of geometric classes in the cohomology of higher-dimensional Kuga-Sato varieties. The main result of this paper relates these diagonal classes to p-adic logarithms of Heegner or Stark-Heegner points, depending on whether K is complex or real respectively.

PREFACE

Over the last three decades, the method of *Euler systems* has been honed into a powerful and versatile technique for relating the arithmetic of a motive to its associated L-function, in the spirit of the conjectures of Deligne, Bloch-Beilinson, Bloch-Kato and Perrin-Riou. Among its most notable successes is the proof of the weak Birch and Swinnerton-Dyer conjecture asserting the equality of the algebraic and analytic rank of an elliptic curve over $\mathbf{Q}$ when the latter invariant is ≤ 1, as well as the finiteness of the associated Shafarevich-Tate group. These statements are particularly striking in the rank one setting, given the dearth of systematic techniques for constructing rational or algebraic points on elliptic curves with direct connections to L-function behaviour.

An important precursor of the Euler System concept is the seminal work of Coates and Wiles [**14**] in the mid 1970's, where certain global cohomology classes constructed from norm-compatible collections of elliptic units in $\mathbf{Z}_p$–extensions of an imaginary quadratic field are used to prove the finiteness of Mordell-Weil groups of elliptic curves with complex multiplication, when the L-function of the associated Grossencharakter does not vanish at its center. The stronger method of Euler systems parlays their *tame deformations*, arising from objects defined over tamely ramified abelian extensions of finite, p-power degree, into an efficient approach for establishing the finiteness of Selmer and Shafarevich-Tate groups in addition to Mordell-Weil groups. The genesis of this approach occurs with the work of Francisco Thaine on circular units [**27**] in the late 1980's, whose inspiration can be traced back even further to Kummer. The subsequent transposition of Thaine's approach to the setting of elliptic units is the basis for Karl Rubin's remarkable strengthening [**25**] of the approach of Coates-Wiles, with dramatic consequences for the finiteness of Shafarevich-Tate groups of elliptic curves with complex multiplication. Kolyvagin's almost simultaneous but independent breakthrough [**23**] exploits Heegner points and their connection with special values of L-series exhibited earlier by Gross and Zagier [**21**] to prove the equality of analytic and algebraic ranks and the finiteness of the Shafarevich-Tate group for *all* (modular) elliptic curves over $\mathbf{Q}$ of analytic rank ≤ 1.

Shortly afterwards, Kazuya Kato [**22**] pioneered an entirely different Euler system approach in which Heegner points are replaced by Beilinson elements in the second K-groups of modular curves—more accurately, by their p-adic deformations arising from norm-compatible systems in towers of modular curves, echoing the theme of

p-adic variation that was already present in the work of Coates and Wiles. Some 20 years later, it was realised that Kato's approach could be profitably adapted to other closely related settings, in which Beilinson elements are replaced by so-called Beilinson-Flach elements [**7**] and diagonal cycles on a triple product of modular curves [**17**], whose p-adic deformations—particularly, those that are germane to the study of the Birch and Swinnerton-Dyer conjecture—are referred to as *generalised Kato-classes* in the articles by Darmon–Rotger ([**19**] and [**20**]) or as (specialisations of) *balanced diagonal classes* in the contributions by Bertolini–Seveso–Venerucci ([**11**] and [**12**]) to this collection. These classes are the key to proving the weak Birch and Swinnerton-Dyer conjecture in analytic rank zero for Mordell–Weil groups of elliptic curves over ring class fields of quadratic fields, both imaginary and real [**18**] (see also [**9**] for a simpler variant to this method, applied in greater generality). For instance, if H is the Hilbert class field of a quadratic field K, then the implication

$$\text{“}L(E/H,1) \neq 0 \implies E(H) \text{ is finite”} \tag{0.1}$$

is known unconditionally via these methods. When K is imaginary, the original pathway to such a result, as described in [**1**], rests crucially on the existence of compatible families of Heegner points, as well as building on the theory of congruences between modular forms and on the p-adic uniformisation of Shimura curves. The route to the same result when K is real quadratic is entirely different and makes no use of the theory of complex multiplication, for the simple but compelling reason that no such theory is currently available in the setting of real quadratic fields.

Extending the theory of complex multiplication to real quadratic fields represents the simplest open case of *Hilbert's twelfth problem* aiming to adapt the Jugendtraum of Kronecker to ground fields other than the rational numbers or CM fields. A systematic attempt was initiated around 2000 to formulate a theory of “real multiplication”, involving p-adic rather than complex analytic objects. The resulting real quadratic analogues of Heegner points, defined in [**15**] in terms of Coleman's theory of p-adic integration, are referred to as *Stark-Heegner points*. They are expected to give rise to a systematic norm-compatible supply of global points (on suitable elliptic curves over $\mathbf{Q}$) defined over ring class fields of real quadratic fields. Because of their strong analogy with Heegner points, they form the basis for a *purely conjectural* extension of the approach of Kolyvagin described in [**1**] for proving (0.1) when K is real quadratic, which is discussed for instance in [**4**].

The article [**3**] introduces a different approach to Stark–Heegner points, by realising them as derivatives of Hida–Rankin p-adic L-functions. This point of view leads to the proof in loc. cit. of the rationality of Stark–Heegner points attached to genus characters of real quadratic fields. It also provides the crucial bridge to connect Stark–Heegner points to generalised Kato classes arising from suitable p-adic families of diagonal cycles. The results of [**2**] can likewise be exploited to make a similar comparison with Heegner points. The explicit comparison between Heegner or Stark-Heegner points and generalised Kato classes, with a view to broadening the scope of the conjecture of Perrin-Riou on rational points on elliptic curves [**24**], is the main goal of this volume.

Comparisons of this type between different Euler systems and Heegner points have a number of fruitful antecedents, among which it may be worthwhile to mention the following:

1. A pioneering early work by Rubin [**26**] examines the global Selmer class arising from the Euler system of elliptic units and finds that the logarithm of such a class is proportional to the *square* of the logarithm of a global point arising from a Heegner point construction. This comparison of elliptic units and Heegner points has intriguing consequences for the construction of rational points on CM elliptic curves via the special values of the Katz p-adic L-function of an imaginary quadratic field.
2. In an attempt to extend Rubin's theorem to elliptic curves without complex multiplication, Bernadette Perrin-Riou conjectured in [**24**] that the p-adic logarithm of the global Selmer class arising from p-adic families of Beilinson elements via Kato's method should likewise be expressed in terms of the square of the logarithm of a Heegner point. This is proved in [**28**] for elliptic curves with multiplicative reduction at p, and in [**8**] in the general case. One of the key ingredients in the latter work are the articles [**6**] and [**5**], the latter of which proposes an alternate approach to Rubin's formula based on special values of p-adic Rankin L-series rather than of the Katz p-adic L-function.
3. The systematic study of "p-adic iterated integrals" undertaken in [**16**] leads to a general conjectural formula relating the p-adic logarithms of generalised Kato classes to certain regulators which are linear combinations with algebraic coefficients of products of two logarithms of global points on elliptic curves. This formula is conceptualised in the framework of a p-adic Birch and Swinnerton-Dyer conjecture in [**10**]. The cases where this conjecture is proved unconditionally (thanks to Heegner points) are an important ingredient in the proof of Perrin-Riou's conjecture described in [**8**].

The present volume collects four interrelated articles, partially motivated by the goal of systematically studying the p-adic logarithm of the balanced diagonal class attached to a pair of weight one theta series of an imaginary (resp. real) quadratic field, and of relating it to the *product* of logarithms of two Heegner (resp. Stark–Heegner) points. More precisely, the first article [**19**] gives an overview of the theory of Stark–Heegner points and of Hida–Rankin p-adic L-functions attached to elliptic curves, and explains the general strategy used to relate Stark–Heegner points to generalised Kato classes. The second article [**20**] studies the problem of the p-adic interpolation of the image of diagonal cycles under the étale Abel–Jacobi map, leading to a 3-variable Λ-adic class in Iwasawa cohomology. It establishes moreover an explicit reciprocity law, connecting this class to a Hida–Garrett–Rankin p-adic L-function attached to a triple of Hida families of cusp forms. The third article [**11**] undertakes the construction of a so-called balanced diagonal class in three variables from a different standpoint, by exploiting the invariant theory of the diagonal embedding of GL_2 into its triple product, combined with the Ash–Stevens theory of p-adic distributions. This analytic approach, formulated in the context of Coleman families of modular forms, lends itself to generalisations to higher groups. It allows to establish an explicit reciprocity law in this context, which is at the base of the results of the subsequent article. In turn the constructions of **[20]** deal more directly with the geometry of diagonal cycles and have been investigated further for example in **[13]**. The fourth article **[12]** gives detailed proofs of the formulae relating the product of the p-adic logarithms of two Heegner points or Stark–Heegner points to the specialisation at the weight $(2, 1, 1)$ of the balanced diagonal class. We refer to the extensive introductions of the various chapters for further details.

At present, the collection of Heegner points on a modular elliptic curve, arising from the combination of modularity and of the theory of complex multiplication, still represents the "gold standard" for understanding the Birch and Swinnerton-Dyer conjecture, particularly in analytic rank one, where the crucial issue of producing non-trivial algebraic points of infinite order on elliptic curves becomes inescapable. By contrast, generalised Kato classes, as well as their forebearers arising from elliptic units make *a priori* only tenuous contact with these central issues, upon which further progress on the Birch and Swinnterton-Dyer conjecture would seem to be crucially dependent. Obtaining tight connections between generalised Kato classes and global points on elliptic curves, such as those proved in this volume, is worthwhile for at least two reasons. Firstly, it seems important to understand the extent to which Selmer classes constructed via a p-adic limiting process are related to "motivic" extensions attached to genuine global points on elliptic curves (or more general algebraic cycles on higher dimensional varieties). The results of the present monograph combine with those of **[26]**, **[28]**, **[8]**, **[16]** and **[10]** to present a coherent picture in the setting of generalised Kato classes arising from diagonal cycles on triple products. Secondly, it lends some theoretical support for the theory of Stark–Heegner points, towards the hope of extending the available constructions of rational points on elliptic curves beyond the theory of Heegner points.

This monograph owes a tremendous debt to the vision of Perrin-Riou, whose conjecture of **[24]** is a basic prototype for the results that are proved here. Perrin-Riou's insights into the connection between Euler systems and p-adic L-functions through her fundamental "dual exponential map in p-adic families" also provides a key ingredient for the proofs of our main results. It is therefore a pleasure to dedicate this collection to Bernadette Perrin-Riou on her 65th birthday.

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