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GENERALIZED CURIE-WEISS MODEL AND QUADRATIC PRESSURE IN ERGODIC THEORY

by Renaud Leplaideur & Frédérique Watbled

Abstract. — We explain the Curie-Weiss model in statistical mechanics within an ergodic viewpoint. More precisely, we simultaneously define in $\{-1,+1\}^{\mathbb{N}}$, on the one hand a generalized Curie-Weiss model within the statistical mechanics viewpoint and on the other hand, the quadratic free energy and quadratic pressure within the ergodic theory viewpoint. We show that there are finitely many invariant measures that maximize the quadratic free energy. They are all dynamical Gibbs measures. Moreover, the probabilistic Gibbs measures for the generalized Curie-Weiss model converge to a determined combination of the (dynamical) conformal measures associated with these dynamical Gibbs measures. The standard Curie-Weiss model is a particular case of our generalized Curie-Weiss model. An ergodic viewpoint over the Curie-Weiss-Potts model is also given.

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Renaud Leplaideur, ISEA, Université de Nouvelle Calédonie, 145, Avenue James Cook – BP R4 98 851 – Nouméa Cedex. Nouvelle Calédonie • LMBA, UMR6205, Université de Brest • *E-mail :* Renaud.Leplaideur@univ-nc.nc • *Url :* http://pagesperso.univ-brest.fr/~leplaide/

Frédérique Watbled, LMBA, UMR 6205, Université de Bretagne Sud, Campus de Tohannic, BP 573, 56017 Vannes, France • *E-mail :* frederique.watbled@univ-ubs.fr

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Résumé (*Généralisation du modèle de Curie-Weiss et Pression quadratique en théorie ergodique*). — On explique ici un modèle généralisé de Curie-Weiss (champ moyen) en utilisant le vocabulaire de la théorie ergodique. On introduit le concept de pression quadratique en théorie ergodique et on montre que pour tout potentiel Hölder dans le sous-shift unilatère $\{-1,+1\}^{\mathbb{N}}$, il n'y a qu'un nombre fini de mesures invariantes qui maximisent la pression quadratique et, que ce sont toutes des mesures d'équilibre pour un multiple du potentiel. On montre que la limite thermodynamique des mesures de Gibbs associées à l'Hamiltonien en champ moyen convergent vers une combinaison des mesures conformes associées à chaque mesure qui maximise la pression quadratique. Le cas standard de Curie-Weiss s'obtient pour un exemple particulier de potentiel. Enfin, le modèle de Curie-Weiss-Potts est également expliqué avec le vocabulaire de la théorie ergodique.

1. Introduction

1.1. Background, main motivations and results. — The notion of Gibbs measure comes from statistical mechanics. It has been studied a lot from the probabilistic viewpoint (see [13, 6, 9, 10]). This notion was introduced in ergodic theory in the 70's by Sinai, Ruelle and Bowen (see [27, 28, 25, 24, 2]). Since that moment, the thermodynamic formalism in dynamical systems became a purely mathematical question and has somehow become isolated from the original physical questions.

It has turned out that this situation has generated sources of confusions. The first one is that while people share the same vocabulary, it is not clear that the same names precisely define the same notions in each viewpoint (ergodic vs physical). We *e.g.* refer to *phase transition, Gibbs measures, pressure*. Furthermore, the confusion is also internal to ergodic theory. Indeed, the thermodynamic formalism is presented very differently for $\mathbb{Z}$-actions (where the transfer operator plays a crucial role) or for $\mathbb{Z}^d$-actions (with $d > 1$). For this later case, the thermodynamic formalism is much closer to what people in statistical mechanics or in probability do. Several questions arising for 1-dimensional actions ergodic theory have to be exported to the higher dimensional case (see [4, 1]). Therefore, it became important to make clear similitudes and differences in the thermodynamic formalism between physical and (1-d) ergodic viewpoints.

Our first result (see Theorem 1.1) states a kind of dictionary between thermodynamic formalism in statistical mechanics and probability on the one hand, and ergodic theory on the other hand. More precisely we explain with the ergodic vocabulary the first-order phase transition arising for the Curie-Weiss model (mean field case), and make precise the link between Gibbs measures within the physical/probabilistic viewpoints and the ergodic viewpoint. We initially decided to focus on the mean field case for the following reasons. First, there is a large amount of literature dealing with this topic. Second, the mean field model is naturally represented into $\{-1,+1\}^{\mathbb{N}}$ and exhibits "physical phase transitions" that we wanted to compare with "1-d ergodic phase transitions" in $\{-1,+1\}^{\mathbb{N}}$.

From there, a subsequent task was to get a similar dictionary for the Curie-Weiss-Potts model which is a generalization of the Curie-Weiss model. This is done in Theorem 1.4.

These two results are then the motivation for our main result (see Theorem 1.3). The key point is that the Hamiltonian for the Curie-Weiss model is almost equal to the square of a Birkhoff sum. The Birkhoff sum is a key object in dynamical systems. We thus introduce within the ergodic viewpoint the notion of *quadratic free energy*. It is equal to the entropy plus the square of an integral. We are naturally led to study a variational principle, distinguishing the invariant measures that maximize the quadratic free energy. This maximum defines the *quadratic pressure*. At the same time, we introduce a generalized Hamiltonian in the Curie-Weiss model and show the link between the associated Gibbs measures (within physical/probabilistic viewpoint) and the Gibbs measures within the ergodic viewpoint. We show how first order phase transitions for this generalized Curie-Weiss model are related to a bifurcation into the set of measures which maximize the quadratic free energy. Theorem 1.1 is thus a particular case of Theorem 1.3.

We believe that this quadratic pressure generates further possible research questions in ergodic theory. Some of them are discussed later (see Subsubsection 1.2.5). Similarly, we believe that our generalized Curie-Weiss model may have physical interest.

Finally, we show that Theorem 1.3 is not an extension of Theorem 1.4. There is no obstruction to defining and studying the quadratic pressure for a more general subshift of a finite type. Nevertheless, the Hamiltonian for the Curie-Weiss-Potts model does not write itself as a square of a Birkhoff sum, because one considers a vector-valued "potential". This is work in progress to give an extension of Theorem 1.4 with the flavour of Theorem 1.3.

1.2. Precise settings and results. —

1.2.1. *Ergodic and Dynamical setting.* — We consider a finite set Λ with a cardinality greater than or equal to 2. It is called the alphabet. We also consider the one-sided full shift $\Sigma = \Lambda^{\mathbb{N}}$ over Λ. A point x in Σ is a sequence $x_0, x_1, \ldots$ (also called an infinite word) where the x_i are in Λ. Most of the time we shall use the notation $x = x_0x_1x_2\ldots$

A $x_i \in \Lambda$ can either be called a letter, or a digit or a symbol.

The shift map σ is defined by

$$\sigma(x_0x_1x_2\ldots) = x_1x_2\ldots.$$

The distance between two points $x = x_0x_1\dots$ and $y = y_0y_1\dots$ is given by

$$d(x, y) = \frac{1}{2^{\min\{n,\ x_n \neq y_n\}}}.$$

A finite string of symbols $x_0 \dots x_{n-1}$ is also called a *word*, of length n. For a word w, its length is $|w|$. A *cylinder* (of length n) is denoted by $[x_0 \dots x_{n-1}]$. It is the set of points y such that $y_i = x_i$ for $i = 0, \dots n-1$. We shall also talk about n-cylinder instead of cylinder of length n.

If w is the word of finite length $w_0 \dots w_{n-1}$ and x is a word, the concatenation wx is the new word $w_0w_1 \dots w_{n-1}x_0x_1 \dots$.

For $\psi : \Sigma \to \mathbb{R}$ continuous and $\beta > 0$, the *pressure function* is defined by

$$\mathcal{P}(\beta\psi) := \sup_{\mu} \left\{ h_\mu + \beta \int_\Sigma \psi \, d\mu \right\}, \tag{1}$$

where the supremum is taken among the set $\mathcal{M}_\sigma(\Sigma)$ of σ-invariant probabilities on Σ and h_μ is the Kolmogorov-Sinaï entropy of μ. The real parameter β is assumed to be positive because it represents the inverse of the temperature in statistical mechanics. It is known that the supremum is actually a maximum and any measure for which the maximum is attained in (1) is called an *equilibrium state for $\beta\psi$*. We refer the reader to [2, 25] for basic notions on thermodynamic formalism in ergodic theory.

If ψ is Lipschitz continuous then the Ruelle theorem (see [23]) states that for every β, there is a unique equilibrium state for $\beta\psi$, which is denoted by $\widetilde{\mu}_{\beta\psi}$. It is *ergodic* and it shall be called the *dynamical Gibbs measure* (DGM for short[1]). It is the unique σ-invariant probability measure which satisfies the property that for every $x = x_0x_1 \dots$ and for every n,

$$e^{-C_\beta} \leq \frac{\widetilde{\mu}_{\beta\psi}([x_0 \dots x_{n-1}])}{e^{\beta . S_n(\psi)(x) - n\mathcal{P}(\beta\psi)}} \leq e^{C_\beta}, \tag{2}$$

where C_β is a positive real number depending only on β and ψ (but not on x or n), and $S_n(\psi)$ stands for $\psi + \psi \circ \sigma + \dots + \psi \circ \sigma^{n-1}$.

In this setting, the *$\beta\psi$-conformal measure* is the unique probability measure such that for every x and for every n,

$$\nu_{\beta\psi}([x_0 \dots x_{n-1}]) = \int e^{\beta S_n(\psi)(x_0 \dots x_{n-1}y) - n\mathcal{P}(\beta\psi)} \, d\nu_{\beta\psi}(y). \tag{3}$$

A precise (and more technical) definition of conformal measure is given in page 207, where the connection between conformal measures and DGM is stated. We emphasize that in our setting, conformal measures and DGM are equivalent measures and one can obtain one from the other.

If the choice of ψ is clear we shall drop the ψ and write $\widetilde{\mu}_\beta$, ν_β and $\mathcal{P}(\beta)$.

1. We prefer the adjective "dynamical" instead of "ergodic" to avoid the discussion if an ergodic Gibbs measure is ergodic or not.

1.2.2. *The Curie-Weiss model.* — We consider the case $\Lambda = \{-1, +1\}$; Σ will be denoted by Σ_2.

If $\omega_0 \dots \omega_{n-1}$ is a finite word, we set

$$H_n(\omega) := -\frac{1}{2n} \sum_{i,j=0}^{n-1} \omega_j \omega_i. \tag{4}$$

It is called the *Curie-Weiss Hamiltonian.* The *empirical magnetization* for ω is $m_n(\omega) := \frac{1}{n} \sum_{j=0}^{n-1} \omega_j$. Then we have

$$H_n(\omega) = -\frac{n}{2}(m_n(\omega))^2. \tag{5}$$

We denote by $\mathbb{P} := \rho^{\otimes \mathbb{N}}$ the product measure on Σ_2, where ρ is the uniform measure on $\{-1, 1\}$, i.e. $\rho(\{1\}) = \rho(\{-1\}) = \frac{1}{2}$, and we define the *probabilistic Gibbs measure* (PGM for short) $\mu_{n,\beta}$ on Σ_2 by

$$\mu_{n,\beta}(d\omega) := \frac{e^{-\beta H_n(\omega)}}{Z_{n,\beta}} \mathbb{P}(d\omega), \tag{6}$$

where $Z_{n,\beta}$ is the normalization factor

$$Z_{n,\beta} = \frac{1}{2^n} \sum_{\omega', \ |\omega'|=n} e^{-\beta H_n(\omega')}.$$

Note that $\mu_{n,\beta}$ can also be viewed as a probability defined on Λ^n.

The measure $\mathbb{P}$ is a Bernoulli measure and is σ-invariant. In ergodic theory it is usually called the Parry-measure (see [21]) and turns out to be the unique measure with maximal entropy. With our previous notations it corresponds to the DGM $\widetilde{\mu}_0$.

If P_n, P are probability measures on the Borel sets of a metric space S, we say that P_n converges weakly to P if $\int_S f\, dP_n \to \int_S f\, dP$ for each f in the class $C_b(S)$ of bounded, continuous real functions f on S. In this case we write $P_n \underset{n\to+\infty}{\overset{w}{\longrightarrow}} P$.

Our first result concerns the weak convergence of the measures $\mu_{n,\beta}$.

THEOREM 1.1 (Weak convergence for the CW model). — *Let ξ_β be the unique point in $[0, 1]$ which realizes the maximum for*

$$\varphi_I(x) := \log(\cosh(\beta x)) - \frac{\beta}{2} x^2.$$

Let $\widetilde{\mu}_b$ be the dynamical Gibbs measure for $b(\mathbb{1}_{[+1]} - \mathbb{1}_{[-1]})$. Then

$$\mu_{n,\beta} \underset{n\to+\infty}{\overset{w}{\longrightarrow}} \begin{cases} \widetilde{\mu}_0 & \text{if } \beta \leq 1, \\ \frac{1}{2}\left[\widetilde{\mu}_{\beta\xi_\beta} + \widetilde{\mu}_{-\beta\xi_\beta}\right] & \text{if } \beta > 1. \end{cases} \tag{7}$$