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DIFFUSION RATE OF WINDTREE MODELS AND LYAPUNOV EXPONENTS

BY CHARLES FOUGERON

ABSTRACT. — Consider a windtree model with several parallel arbitrary right-angled obstacles placed periodically on the plane. We show that its diffusion rate is the largest Lyapunov exponent of some stratum of quadratic differentials and exhibit a new general strategy to compute the generic diffusion rate in a family of such models. This result enables us to numerically compute the diffusion rates of a wide class of windtree models and to observe its asymptotic behavior according to the shape of the obstacles.

RÉSUMÉ (*Diffusion du vent dans les arbres et exposants de Lyapunov*). — Nous considérons un modèle de vent dans les arbres avec des obstacles reproduits périodiquement dans le plans. Les obstacles seront ici des polygones à angles droits dont un des côtés est parallèle au coté d'un autre obstacle. En introduisant une stratégie générale, nous montrons que le taux de diffusion pour un élément générique de cette famille de modèles est le plus grand exposant de Lyapunov associé à une strate de différentielles quadratiques. Celui permet un calcul numérique des taux de diffusion sur une grande variétés de modèles et nous observons dans un deuxième temps le comportement assymptotique de celui-ci en faisant varier la forme des obstacles.

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1. Introduction

The windtree model was first introduced by Paul and Tatiana Ehrenfest in 1912 [7] as part of statistical physics investigations. In this book they set a simplified model for non-interacting light particles moving around massive particles that do not move, but on which the light particles collide with elastic collisions. We classically refer to the light particles as the *wind* and the static ones as *trees*. The motivation of the two physicists was to understand the kinetic behavior of such a system. They asked, among others, the following question: *for a generic disposition of square trees orientated in the same direction, does the speed of K light particles equidistribute asymptotically in the four possible directions?*

Plenty of questions have been studied on this model, in particular for the $\mathbb{Z}^2$-periodic case with square obstacles. The results feature alternatively elements of chaotic and periodic behavior. In [16] the recurrence of billiard flow was proven along with abnormal diffusion for special dimensions of the obstacles. In [12] the genericity of non-ergodic behavior was shown, and its diffusion rate was computed to be $2/3$ in [5]. A positive answer to the original question has only been provided very recently by [17].

In parallel a similar model with smooth convex obstacles has been studied by a large amount of mathematicians throughout the twentieth century (see e.g. [2] or [19]). In this case, the billiards satisfy some hyperbolicity property and the behavior of its flow is closely related to a Brownian motion.

A good tool to check if a polygonal windtree model has such an hyperbolic behavior is provided by the diffusion rates which should be $1/2$ in the case of Brownian-like motions. In particular the result of [5] destroys any hope of directly applying the methods of the smooth convex case to the rectangular model. The question is still open in the case of the asymptotic behavior of polygonal shapes approaching smooth convex ones, for example with the circle: is the diffusion rate of periodic windtree models with regular n-gons going to $1/2$ when n goes to ∞? We hope that developing methods to compute these diffusion rates in more general settings will provide a first step to understanding this asymptotic behavior and the non-convex obstacles cases.

The arguments of [5] rely on a remarkable correspondence between the diffusion rate of an infinite periodic billiard table and the Lyapunov exponent of an associated translation surface. This computation was generalized in [6] to any $\mathbb{Z}^2$-periodic windtree whose trees have only right angles and are horizontally and vertically symmetric. In every of these cases, the corresponding Lyapunov exponent belongs to some 2 dimensional subbundle of the Hodge bundle. Moreover in all of these cases the Lyapunov exponent is rational and can be computed using some geometric arguments.

We introduce a general strategy to exhibit the Lyapunov exponent of some locus in a stratum that correspondsto the diffusion rate of a given periodic windtree model. First we identify a common orbit closure of almost all translation surfaces associated to a family of windtree tables; then we find an irreducible subbundle of the Hodge bundle on this locus whose top Lyapunov exponent is exactly the diffusion rate. The tools for the first craft are given by recent results of [10], [21] and [22] and are introduced in subsection 4.2. For the second one, we show an additional lemma to the work of [3] which yields the diffusion rate for any translation surface in a generic direction.

In particular, we show that computing the orbit closure of a generic element of a family of windtree models boils down to constructing good examples in the family, for which there are favorable cylinders. This enables us to show inductively that the initial orbit closure contains bigger and bigger families of translation surfaces, and eventually to prove the density of almost all surfaces in the family.

We apply this method to the case of a periodic windtree with several obstacles in its fundamental domain. We pick a family of $n \geq 2$ rectangular obstacles inside a rectangle, each rectangle having its sides parallel to one side of the fundamental rectangle, and repeat this table $\mathbb{Z}^2$-periodically in the plane. We can then show the following theorem,

THEOREM 1.1. — *For all $n \geq 2$, and almost every length parameter in this family of windtree models, in almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n})$.*

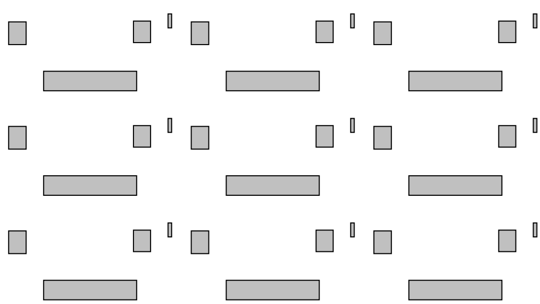

FIGURE 1.1. An example of a configuration of obstacles for which Theorem 1.1 applies for generic lengths

This theorem generalizes to obstacles with an arbitrary number of right angles, whose sides are parallel to the side of the rectangular fundamental domain. If n is the number of obstacles and p the total number of inward (concave) right angles in all the obstacles, we have a similar result,

THEOREM 1.2. — *For all $n \geq 2$, $p \geq 0$, and almost every length parameter in this family of windtree models, in almost every direction, the diffusion rate is equal to the top Lyapunov exponent of $\mathcal{Q}(1^{4n+p}, -1^p)$.*

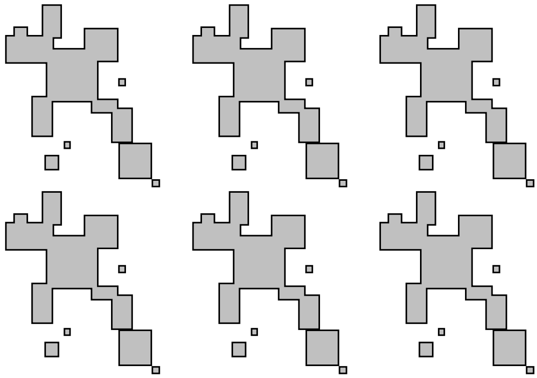

FIGURE 1.2. An example of a configuration of obstacles for which Theorem 1.2 applies for generic lengths

In the last section we discuss the value of these exponents by running numerical experiments with a Sage code developed by the author in a collaborative project [4]. These experiments give strong evidence that the family we have introduced above can approach arbitrarily close to any diffusion rate between 1/2 and 0. In particular it goes to 1/2 (*i.e.* the diffusion rate of the Brownian motion) when the number of obstacles goes to infinity.

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2. Translation surfaces

2.1. Definitions. — A *translation surface* is a surface whose change of charts are translations. Such a surface is endowed with a flat metric (the pull-back of the canonical metric on $\mathbb{R}^2$) and a canonical direction.

One way to think of these translation surfaces is by gluing the sides of a polygon via translations. Let P be a polygon with $2k$ edges and let $z_1, \ldots, z_{2k}$ be complex numbers associated with the vectors of its sides. We assume that $z_i = z_{k+i}$, and glue the sides z_i and z_{k+i} to obtain a flat surface with conical singularities of angle multiples of 2π. These numbers are called *periods* of the translation surface.

We can define similar structures allowing the change of charts to be also translations composed with $-\mathrm{Id}$. The class of surfaces we obtain are called *half-translation* surfaces. The periods are still defined in the same way, but depend up to a sign on the choice of side.

Using triangulations Veech showed in [20] that this is a general construction with a notion of *pseudo-polygons* (in a much wider class of structures). The complex numbers $(z_i)_{1\leq i\leq k}$ (defined up to a sign in the case of half-translation surfaces) induce local coordinates in the moduli space of such structures, we call them *period coordinates*. We will introduce them as periods of abelian differentials below.

2.1.1. *Differentials and moduli spaces.* — There is a one-to-one correspondence between compact translation surfaces and Riemann surfaces equipped with a non-zero holomorphic 1-form. As well as between compact half-translation surfaces and Riemann surfaces equipped with quadratic differentials.

For $g \geq 1$ let $d_1, \dots, d_k \geq 0$ and $m_1, \dots, m_k \geq -1$ be integers such that $d_1 + \cdots + d_k = 2g-2$ and $m_1 + \cdots + m_k = 4g-4$. The strata $\mathcal{H}(d_1, \dots, d_k)$ and $\mathcal{Q}(m_1, \dots, m_k)$ are defined to be the sets of couples (S, ω) and (S, q) where S is a genus g closed Riemann surface, ω is a holomorphic 1-form on S, q is a quadratic differential form eventually with simple poles, and their zeros multiplicities are given respectively by $d_1, \dots, d_k$ and $m_1, \dots, m_k$ (a multiplicity of -1 corresponding to a simple pole). The conical points in a translation surface correspond to the zeros of the differential. If d is the multiplicity of the zero, the angle is equal to $2(d+1)\pi$ (and $(d+2)\pi$ for half-translation surfaces).

Given a translation surface (S, ω), let $\Sigma \subset S$ be the set of zeros of ω. Pick a basis $\{\xi_1, \dots, \xi_n\}$ for the relative homology group $H_1(S, \Sigma; \mathbb{Z})$. The map $\Phi : \mathcal{H}(\alpha) \to \mathbb{C}^n$ defined by

$$\Phi(S, \omega) = \left(\int_{\xi_1} \omega, \dots, \int_{\xi_n} \omega \right)$$

redefines local period coordinates with translation as change of charts as above.

There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on connected components of strata coming from the linear action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$ in charts. For any translation surface in a stratum, its orbit closure via this action is some affine invariant manifold of the stratum: it is defined in local period coordinates by linear equations. They are endowed with a canonical measure supported on these manifolds called affine measures [9], [10].

In these coordinates, and for any affine subspace, we can define a notion of zero Lebesgue measure subsets. This is what we will refer to when saying Lebesgue-almost every surface. It can also be understood with respect to the Masur–Veech measure in the whole stratum (see [24] for a detailed introduction).

2.1.2. *Translation cover.* — To any half-translation surfaces S which is not a squared holomorphic form we associate its *translation cover* $\hat{S}$ corresponding to the subgroup of the fundamental group with holonomy equal to -1. It is a double cover. We endow $\hat{S}$ with the pulled-back metric of S which defines a translation surface structure for $\hat{S}$.