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EXPLICIT LINEARIZATION OF ONE-DIMENSIONAL GERMS THROUGH TREE-EXPANSIONS

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Abstract. — We explain Écalle's "arbomould formalism" in its simplest instance, showing how it allows one to give explicit formulas for the operators naturally attached to a germ of holomorphic map in one dimension. When applied to the classical linearization problem of non-resonant germs, which contains the well-known difficulties due to the so-called small divisor phenomenon, this elegant and concise tree formalism yields compact formulas, from which one easily recovers the classical analytical results of convergence of the solution under suitable arithmetical conditions on the multiplier. We rediscover this way Yoccoz's lower bound for the radius of convergence of the linearization and can even reach a global regularity result with respect to the multiplier (C^1-holomorphy) which improves on Carminati-Marmi's result.

Résumé (*Linéarisation explicite de germes unidimensionnels par développement en arbres*). — Nous expliquons le formalisme des « arbomoules » d'Écalle dans le cas le plus simple et montrons comment il permet d'obtenir des formules explicites pour les opérateurs naturellement associés à un germe d'application holomorphe en une dimension. Dans le cadre du problème classique de la linéarisation des germes non résonants, qui contient la difficulté bien connue due au phénomène des petits diviseurs, ce formalisme élégant et concis reposant sur des arbres fournit des formules compactes, dont on déduit aisément les résultats analytiques classiques de convergence de la solution

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moyennant des conditions arithmétiques appropriées sur le multiplicateur. Nous retrouvons de cette façon la borne inférieure due à Yoccoz pour le rayon de convergence de la linéarisation et obtenons même un résultat de régularité globale par rapport au multiplicateur (C^1-holomorphie) qui améliore un résultat de Carminati et Marmi.

1. Introduction

The purpose of this article is twofold:

– to expound J. Écalle's "arbomould formalism" by illustrating it on the linearization problem for holomorphic germs in one complex dimension—this amounts to a novel approach to *formal* linearization by means of a powerful and elegant combinatorial machinery,
– to show how this allows one to find again the classical *analytic* results on the convergence of the formal linearization by Kœnigs, Siegel, Bruno, Yoccoz, and even improve on Carminati-Marmi's result on the regularity with respect to the multiplier.

The classification of diffeomorphisms near fixed points is one of the starting points for Poincaré's theory of normal forms and has its roots in 19^{th} century mathematics, with E. Schröder and G. Kœnigs's works on the linearization problem in one complex dimension (see e.g., [25]). The problem consists in finding a conjugacy between a map $g\colon\ z \mapsto qz + O(z^2)$ holomorphic near the origin and its linear part $z \mapsto qz$, assuming that the multiplier q is non-zero. One thus looks for an invertible map $z \mapsto h(z)$ such that $g \circ h(z) = h(qz)$ (i.e., the inverse of h should satisfy the Schröder functional equation). At a formal level, there is a solution as soon as q is not a root of unity, i.e., there is a formal linearization $h(z) \in \mathbb{C}[[z]]$ in that case, and the Kœnigs linearization theorem asserts its convergence whenever $|q| \neq 1$.

When $|q| = 1$, things are much more delicate because the recursive expressions available for the coefficients of the formal linearization h involve denominators of the form $q^n - 1$ with any $n \geq 1$. In the so-called resonant case, namely when q is a root of unity, linearization is in fact generically not even possible at the level of formal series, and the classification of resonant diffeomorphisms leads to Gevrey divergent series and questions of summability and resurgence (see e.g., [10]). For $q = \mathrm{e}^{2\pi\mathrm{i}\omega}$ with ω real and irrational, we are faced with the so-called "small divisor problem", because of the arbitrary smallness of the quantities $q^n - 1$ present in the denominators, and this calls for a suitable number-theoretic hypothesis on ω in order to prove that $h(z)$ is analytic, as shown by H. Cremer, C. L. Siegel, A. D. Bruno and J.-C. Yoccoz.

The article [10] proposed a totally new approach to deal with general singularities of analytic dynamical systems with discrete or continuous time, in any dimension, with an array of techniques to cover the most general situations where the complications due to resonances and small denominators coexist, but this work has not really been assimilated by the dynamical systems community. In that article, J. Écalle introduced the key concept of "arborification," according to which the formal series first expressed as "mould expansions" have to be reencoded by expansions over families of trees.

In the present paper, we explain the basics of Écalle's tree formalism and show how it leads to an explicit formula for the conjugacy $h(z)$. Writing the Taylor expansion of the holomorphic germ g in the form $g(z) = q(z + a_1 z^2 + a_2 z^3 + \cdots)$, we shall obtain

$$h(z) = z + \sum_T \gamma_T \Bigg(\prod_{\sigma \in V_T} \frac{a_{N_T(\sigma)}}{q^{\|\mathrm{Tree}(\sigma,T)\|} - 1} \Bigg) z^{\|T\|+1},$$

where the summation is performed over trees T whose vertices σ are decorated by positive integers $N_T(\sigma)$ and the coefficients γ_T are non-negative rational numbers to be defined in due time; the product is over all the vertices of the given tree T, the notation $\|\cdot\|$ indicating the sum of the decorations of a tree and $\mathrm{Tree}(\sigma, T)$ denoting the subtree of T "rooted at σ".

In fact, Écalle's formalism will give more: it is the composition operator itself $\varphi \in \mathbb{C}[[z]] \mapsto \varphi \circ h \in \mathbb{C}[[z]]$ which can be represented as the sum of a formally summable family of explicit elementary differential operators. This is related to the idea, due to A. Cayley [7], that trees are the relevant combinatorial objects to deal with the composition of differential operators.(1)

All the precise definitions are given below in a self-contained way, and it is in fact one of the objectives of the present text to clarify the notions introduced by Écalle, connecting them with well-known combinatorial objects and constructions, proposing on the way quite a few innovations in the notations and the presentation of the concepts with respect to to the existing literature. An originality of our presentation is that we arrive directly at the tree representation of h or of its composition operator, without constructing a preliminary mould expansion and then passing through the process of arborification. In this sense, the first part of the paper (Sections 3–6) can be considered as a lightened introduction to Écalle's formalism, and the interested reader can pursue with [12] and the references therein to learn more about the algebraic structures underlying arborification.

Next, in the second part of the paper (Sections 7–11), we show that the explicit expression of h obtained in the first part can be efficiently used to prove

1. The use of trees in small divisor problems can be traced back to Eliasson's 1988 preprint, published as [11], containing the direct proof of the convergence of the Lindstedt series in the (more difficult) context of KAM theory. Let us also mention that, for the multidimensional Siegel problem, another tree formalism was used in [3], whose formulas (4.12) and (4.13) are reminiscent of the formula obtained by Écalle's tree formalism.

its analyticity when $q = \mathrm{e}^{2\pi\mathrm{i}\omega}$ and ω satisfies Bruno's arithmetic condition (relying on an arithmetical lemma due to Davie, as in [4]), finding again Yoccoz's lower bound for the radius of convergence of h.

It also gives us access to a new result on the *monogenic dependence* of h with respect to the multiplier q, in the spirit of [15]. The first result of that kind was proved by C. Carminati and S. Marmi in [20]. The idea consists in considering all the ω's satisfying a uniform Bruno condition and constructing a closed subset K of $\mathbb{C}$ such that the map $q \in K \mapsto h \in B$ is C^1-holomorphic, where B is a suitable Banach space of functions of z. When it comes to C^1-holomorphy, our method is quite different from that of [20] and gives an improvement for the radius of the disk in the z-plane which determines the Banach space B that one can take.

We tried to make the paper as self-contained as possible and hope it will constitute an accessible entry to some of the beautiful and far reaching constructions of Écalle, while yielding original proofs of non trivial dynamical results and paving the way for further works.

2. Linearization of diffeomorphisms in dimension 1

Let us review the dynamical setting and fix some notation. We denote by

$$\tilde{\mathscr{G}} := \Big\{ g(z) = \sum_{n\geq 1} b_n z^n \in \mathbb{C}[[z]] \mid b_1 \neq 0 \Big\} \tag{1}$$

the group of formal diffeomorphisms in one dimension, the group operation being the composition of formal series without constant term, with the notation $g^{\circ(-1)}$ for the inverse of an element g. The coefficient b_1 of a given $g \in \tilde{\mathscr{G}}$ is called its *multiplier*; the formal diffeomorphisms with multiplier 1 form a subgroup of $\tilde{\mathscr{G}}$ that we denote by $\tilde{\mathscr{G}}_1$. The group of germs of holomorphic diffeomorphisms in one dimension can be identified with a subgroup $\mathscr{G}$ of $\tilde{\mathscr{G}}$, in which tangent-to-identity convergent series also form a subgroup:

$$\mathscr{G} := \big\{ g \in \tilde{\mathscr{G}} \mid g \in \mathbb{C}\{z\} \big\}, \qquad \mathscr{G}_1 := \mathscr{G} \cap \tilde{\mathscr{G}}_1. \tag{2}$$

The local theory of holomorphic dynamics is concerned with the iteration of elements of $\mathscr{G}$ and the description of the conjugacy classes of $\mathscr{G}$. The rotations $R_q \in \mathscr{G}$ defined by

$$R_q(z) := qz, \qquad \text{for } q \in \mathbb{C}^* \tag{3}$$

display the simplest possible dynamics: the kth iterate of R_q is R_{q^k} (for any $k \in \mathbb{Z}$). One is thus interested in the

Holomorphic Linearization Problem. — *Given $g \in \mathscr{G}$, find $h \in \mathscr{G}_1$ such that*

$$g \circ h = h \circ R_q, \tag{4}$$

where q is the multiplier of g.

It is indeed clear that, if h solves (4), then q cannot be anything else but $\frac{\mathrm{d}g}{\mathrm{d}z}(0)$, and the kth iterate of g is thus $h \circ R_{q^k} \circ h^{\circ(-1)}$. Notice that there is no loss of generality in imposing a priori $h \in \mathscr{G}_1$: if $h_* \in \mathscr{G}$ is a solution of (4) with multiplier λ, then $h_* \circ R_\lambda^{\circ(-1)}$ is a solution which belongs to $\mathscr{G}_1$.

Similarly, we may consider the

Formal Linearization Problem. — *Given $g \in \tilde{\mathscr{G}}$, find $h \in \tilde{\mathscr{G}}_1$ which solves* (4).

A solution h to this problem will be called a *formal linearization of g*. Of course, if $g \in \mathscr{G}$, then a solution of the Formal Linearization Problem with non-zero radius of convergence is the same thing as a solution of the Holomorphic Linearization Problem.

Viewing $\tilde{\mathscr{G}}$ as a skew-product $\mathbb{C}^* \times \tilde{\mathscr{G}}_1$, we will systematically write g in the form

$$g = R_q \circ f, \qquad q \in \mathbb{C}^*, \quad f \in \tilde{\mathscr{G}}_1, \tag{5}$$

so that Equation (4) takes the form $f \circ h = R_q^{\circ(-1)} \circ h \circ R_q$. We first recall the elementary

Lemma 2.1. — *Let $f(z) = z + \sum_{n\geq 1} a_n z^{n+1} \in \tilde{\mathscr{G}}_1$ and $q \in \mathbb{C}^*$. Suppose that q is not a root of unity. Then the Formal Linearization Problem for $g = R_q \circ f$ has a unique solution*

$$h(z) = z + \sum_{n\geq 1} c_n z^{n+1} \in \tilde{\mathscr{G}}_1. \tag{6}$$

The coefficients of the formal linearization are inductively determined by the formula

$$c_n = \frac{1}{q^n - 1} \sum_{r=1}^{n} \sum_{\substack{(n_0,\ldots,n_r)\in\mathbb{N}^{r+1} \ s.t.\\ n_0+\cdots+n_r+r=n}} a_r c_{n_0} \cdots c_{n_r}, \qquad n \geq 1, \tag{7}$$

with the convention $c_0 = 1$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Proof. — Write the conjugacy Equation (4) as $h(qz) = qf\big(h(z)\big)$ and expand it. □