Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON THE ORBITAL STABILITY OF A FAMILY OF TRAVELLING WAVES FOR THE CUBIC SCHRÖDINGER EQUATION ON THE HEISENBERG GROUP

Louise Gassot

Tome 149
Fascicule 1

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 15-54

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 1, tome 149, mars 2021

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
149 (1), 2021, p. 15-54

ON THE ORBITAL STABILITY OF A FAMILY OF TRAVELLING WAVES FOR THE CUBIC SCHRÖDINGER EQUATION ON THE HEISENBERG GROUP

BY LOUISE GASSOT

ABSTRACT. — We consider the focusing energy-critical Schrödinger equation on the Heisenberg group in the radial case

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2+y^2)\partial_s^2, \quad (t,x,y,s) \in \mathbb{R}\times\mathbb{H}^1,$$

which is a model for non-dispersive evolution equations. For this equation, the existence of global smooth solutions and the uniqueness of weak solutions in the energy space are open problems. We are interested in a family of ground-state travelling waves parameterized by their speed in $(-1,1)$. We show that the travelling waves of speed close to 1 present some orbital stability in the following sense. If the initial data is radial and close enough to one traveling wave, then there exists a global weak solution that stays close to the orbit of this travelling wave at all times. A similar result is proven for the limiting system associated to this equation.

Texte reçu le 30 septembre 2019, modifié le 9 septembre 2020, accepté le 2 octobre 2020.

LOUISE GASSOT, Département de mathématiques et applications, École normale supérieure, CNRS, PSL University, 75005 Paris, France, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405 Orsay, France • *E-mail :* `louise.gassot@universite-paris-saclay.fr`

Mathematical subject classification (2010). — 35B35, 35C07, 35Q55, 43A80.

Key words and phrases. — Nonlinear Schrödinger equation, Traveling wave, Orbital stability, Heisenberg group, Dispersionless equation, Bergman kernel.

The author received no specific funding for this work.

 0037-9484/2021/15/$ 5.00
 doi:10.24033/bsmf.2824

Résumé (*Autour de la stabilité orbitale d'une famille d'ondes progressives pour l'équation de Schrödinger cubique sur le groupe de Heisenberg*). — On considère l'équation de Schrödinger énergie-critique sur le groupe de Heisenberg dans le cas radial

$$i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u, \quad \Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2+y^2)\partial_s^2, \quad (t,x,y,s) \in \mathbb{R}\times\mathbb{H}^1,$$

qui est un modèle d'équation d'évolution non dispersive. Pour cette équation, dans l'espace d'énergie, l'existence globale de solutions régulières et l'unicité de solutions faibles sont des problèmes ouverts. On s'intéresse à une famille d'ondes progressives minimisantes paramétrées par leur vitesse dans $]-1,1[$. On montre que les ondes progressives dont la vitesse est proche de 1 possèdent des propriétés de stabilité orbitale au sens suivant. Pour toute donnée initiale radiale suffisamment proche d'une onde progressive, alors il existe une solution faible globale associée à cette donnée initiale qui reste proche de l'orbite de l'onde progressive en tout temps. Un résultat similaire est montré pour le système limite associé à cette équation.

1. Introduction

1.1. Motivation. — We are interested in the Schrödinger equation on the Heisenberg group

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}^1} u = |u|^2 u \\ u(t=0) = u_0 \end{cases}, \quad (t,x,y,s) \in \mathbb{R}\times\mathbb{H}^1. \tag{1}$$

The operator $\Delta_{\mathbb{H}^1}$ denotes the sub-Laplacian on the Heisenberg group. When the solution is radial, in the sense that it only depends on t, $|x+iy|$ and s, the sub-Laplacian writes as

$$\Delta_{\mathbb{H}^1} = \frac{1}{4}(\partial_x^2 + \partial_y^2) + (x^2+y^2)\partial_s^2.$$

The Heisenberg group is a typical case of sub-Riemannian geometry where dispersive properties of the Schrödinger equation disappear (see Bahouri, Gérard and Xu [3]). To take it further, Del Hierro [12] proved sharp decay estimates for the Schrödinger equation on H-type groups, depending on the dimension of the center of the group. More generally, Bahouri, Fermanian and Gallagher [2] proved optimal dispersive estimates on stratified Lie groups of step 2 under some property of the canonical skew-symmetric form. In contrast, they also gave a class of groups without this property displaying a total lack of dispersion, which includes the Heisenberg group.

Dispersion impacts the way one can address the Cauchy problem for the Schrödinger equation. Indeed (see Burq, Gérard and Tzvetkov [7], Remark 2.12), the existence of a smooth local in the time-flow map defined on some Sobolev space $H^k(M)$ for the Schrödinger equation on a Riemannian manifold M with the Laplace–Beltrami operator Δ

$$\begin{cases} i\partial_t u - \Delta u = |u|^2 u \\ u(t=0) = u_0 \end{cases}$$

implies the following Strichartz estimate

$$\|e^{it\Delta} f\|_{L^4([0,1]\times M)} \leq C\|f\|_{H^{\frac{k}{2}}(M)}.$$

The argument also applies for the Heisenberg group with the homogeneous Sobolev spaces $\dot{H}^k(\mathbb{H}^1)$, for which the inequality holds if and only if $k \geq 2$ [9]. In particular, without a conservation law controlling the $\dot{H}^2$-norm, there is no existence result of global smooth solutions. Moreover, existence and uniqueness of weak solutions in the energy space $\dot{H}^1(\mathbb{H}^1)$ is an open problem, even if constructing global weak solutions to the Schrödinger equation on the Heisenberg group would still possible in the defocusing case through a compactness argument. Note that for weak solutions the energy of the solution is only bounded above by the initial energy. Therefore, the cancellation of the energy of the solution at some time may not imply that the solution is identically zero and does not exclude the possibility of non-uniqueness of weak solutions, as in the 2D incompressible Euler equation [15].

The aim of this paper is to construct some global weak solutions with a prescribed behaviour. More precisely, given initial data close to some ground-state travelling wave solution for the Schrödinger equation on the Heisenberg group, we want to construct a global weak solution that stays close to the orbit of the traveling wave at all times. Combined with a uniqueness result, this would lead to the orbital stability of this ground-state traveling wave.

1.2. Main results. — We consider a family of traveling waves with speed $\beta \in (-1,1)$ in the form

$$u_\beta(t,x,y,s) = \sqrt{1-\beta}Q_\beta(x,y,s+\beta t).$$

The profile Q_β satisfies the following stationary hypoelliptic equation (with $D_s = -i\partial_s$)

$$-\frac{\Delta_{\mathbb{H}^1} + \beta D_s}{1-\beta} Q_\beta = |Q_\beta|^2 Q_\beta.$$

Because of the scaling invariance, it would have been equivalent in the rest of the study to define u_β as

$$u_\beta(t,x,y,s) = Q_\beta\left(\frac{x}{\sqrt{1-\beta}}, \frac{y}{\sqrt{1-\beta}}, \frac{s+\beta t}{1-\beta}\right).$$

From [8], we know that as β tends to 1, the ground-state solutions of speed β converge up to symmetries in $\dot{H}^1(\mathbb{H}^1)$ to some profile Q. Moreover, Q is the