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THE ARNOUX–YOCCOZ MAPPING CLASSES VIA PENNER'S CONSTRUCTION

by Livio Liechti & Balázs Strenner

Abstract. — We give a new description of the Arnoux–Yoccoz mapping classes as a product of two Dehn twists and a finite order element. The construction is analogous to Penner's construction of mapping classes with small stretch factors.

Résumé (*Les homéomorphismes de Arnoux–Yoccoz via la construction de Penner*). — Nous donnons une nouvelle description des homéomorphismes de Arnoux–Yoccoz comme un produit de deux twists de Dehn et d'un élément d'ordre fini. Cette construction est analogue à celle des homéomorphismes pseudo-Anosovs de petite dilatation donnée par Penner.

1. Introduction

The mapping class group of a surface S is the group of isotopy classes of orientation-preserving homeomorphisms of S. Motivated by studying geometric structures on 3-manifolds, Thurston [24] modernized the theory of mapping class groups in the 1970s by giving a classification of elements into three types:

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finite order, reducible and pseudo-Anosov. This article concerns the third type. A mapping class is *pseudo-Anosov* if it has a representative homeomorphism ϕ and singular measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ on S such that $\phi(\mathcal{F}^u) = \lambda\mathcal{F}^u$ and $\phi(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$ for some $\lambda > 1$. The number λ is independent of choice of representative homeomorphism, and it is called the *stretch factor* or *dilatation* of the pseudo-Anosov mapping class.

Thurston showed that stretch factors of pseudo-Anosov mapping classes of the closed orientable surface S_g are algebraic integers with degree bounded above by $6g-6$. He claimed without proof in [24] that the degree $6g-6$ was realizable, but this statement was only recently proven in [22]. For some time, however, even the fact that pseudo-Anosov stretch factors of arbitrarily large degrees exist was not justified. This fact was first shown by Arnoux and Yoccoz [4] in 1981. They constructed a pseudo-Anosov mapping class $\tilde{h}_g$ on S_g for each $g \geq 3$ with a stretch factor of algebraic degree g. After stating the main results, we will recall the construction in Section 2.1 and give more reasons for why the Arnoux–Yoccoz examples are of importance.

Despite the mapping classes $\tilde{h}_g$ probably being the single most widely studied explicit family of pseudo-Anosov mapping classes, to this day, no constructions have been known other than the original approach by Arnoux and Yoccoz.

The goal of this paper is to present a new description as a product of two Dehn twists and a finite order mapping class. We hope that this new description will shed new light on the examples and help construct new analogous families of mapping classes that might also serve as interesting examples. An alternative description of the Arnoux–Yoccoz mapping classes was also asked for by Margalit in Section 10 of [16].

Theorem 1.1. — *The Arnoux–Yoccoz mapping class $\tilde{h}_g$ on the surface S_g is conjugate to $\tilde{f}_g = r \circ T_a \circ T_b^{-1}$, where T_a and T_b^{-1} are positive and negative Dehn twists about the curves a and b pictured on Figure 1.1, and r is a rotation of the surface by one click in either direction.*

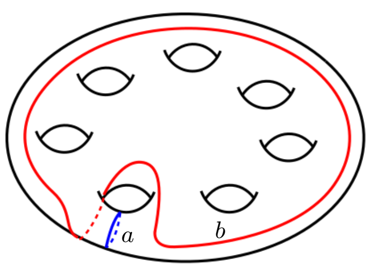

Figure 1.1. The surface S_g with a rotational symmetry of order g. This figure shows the case $g = 7$.

For the proof, we use the fact, shown by the second author in [23, Section 5], that the mapping classes $\tilde{h}_g$ arise as lifts of mapping classes on nonorientable surfaces. More precisely, there is a pseudo-Anosov mapping class h_g (see Section 2.2 for the definition) on the closed nonorientable surface N_{g+1} of genus $g+1$ for each $g \geq 3$, such that $\tilde{h}_g$ is the lift of h_g by the double cover $S_g \to N_{g+1}$. We will deduce Theorem 1.1 from the following.

Theorem 1.2. — *The nonorientable Arnoux–Yoccoz mapping class h_g on the surface N_{g+1} is conjugate to $f_g = r \circ T_c$, where T_c is a Dehn twist about the two-sided curve c pictured in Figure 1.2, and r is a rotation of the surface by one click in either direction.*

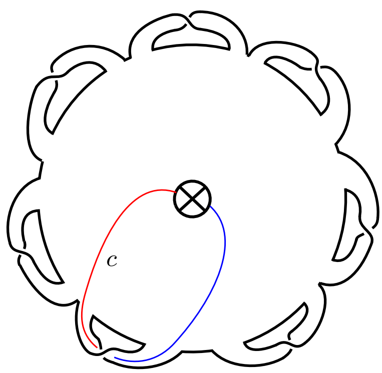

Figure 1.2. The circle with an X inside it indicates a crosscap: the inside of the circle is not part of the surface and antipodal points of the circle are identified. A disk with one crosscap is, therefore, a Möbius strip. So, this figure shows a nonorientable surface obtained by attaching g twisted bands to the boundary of a Möbius strip. The surface has one boundary component. By gluing a disk to the boundary component, we obtain the closed surface N_{g+1}.

The direction of twisting about T_c is important, see Figure 3.2 later for the reason behind this. On a nonorientable surface, there is no notion of positive or negative twisting, so we specify the direction of the Dehn twist T_c by the coloring of the curve c on Figure 1.2 as follows. By cutting out the crosscap in the middle and cutting the twisted bands, we obtain an orientable surface with an embedding in $\mathbf{R}^2$ coming from the figure. Our cut-up surface inherits the orientation of $\mathbf{R}^2$. The blue and red parts of our curve indicate the parts where the twisting behaves like a positive and negative twist, respectively, with respect to the orientation of the cut-up surface.