

# Bulletin

de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Loïc Poulain d'Andecy & Salim Rostam

Tome 149  
Fascicule 1

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 179-233

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*Le Bulletin de la Société Mathématique de France* est un périodique trimestriel  
de la Société Mathématique de France.

Fascicule 1, tome 149, mars 2021

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***Tarifs***

*Vente au numéro : 43 € (\$ 64)*

*Abonnement électronique : 135 € (\$ 202),*

*avec supplément papier : Europe 179 €, hors Europe 197 € (\$ 296)*

Des conditions spéciales sont accordées aux membres de la SMF.

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*Bulletin de la Société Mathématique de France*

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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

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## MORITA EQUIVALENCES FOR CYCLOTOMIC HECKE ALGEBRAS OF TYPES B AND D

BY LOÏC POULAIN D'ANDECY & SALIM ROSTAM

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**ABSTRACT.** — We give a Morita equivalence theorem for so-called cyclotomic quotients of affine Hecke algebras of types B and D, in the spirit of a classical result of Dipper–Mathas of type A for Ariki–Koike algebras. Consequently, the representation theory of affine Hecke algebras of types B and D reduces to the study of their cyclotomic quotients with eigenvalues in a single orbit under multiplication by  $q^2$  and inversion. The main step in the proof consists in a decomposition theorem for generalisations of quiver Hecke algebras that recently appeared in the study of affine Hecke algebras of types B and D. This theorem reduces the general situation of a disconnected quiver with involution to a simpler setting. To be able to treat types B and D at the same time we unify the different definitions of quiver Hecke algebra for type B that exist in the literature.

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*Texte reçu le 23 septembre 2019, modifié le 4 septembre 2020, accepté le 20 octobre 2020.*

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Mathematical subject classification (2010). — 20C08.

Key words and phrases. — Cyclotomic Hecke algebra, Morita equivalence, Quiver Hecke algebras, Representation theory.

The first author is supported by *Agence Nationale de la Recherche* through the JCJC project ANR-18-CE40-0001.

RÉSUMÉ (*Équivalences de Morita pour les algèbres de Hecke cyclotomiques de type B et D*). — Nous énonçons un théorème d'équivalence de Morita pour les quotients cyclotomiques des algèbres de Hecke affines de type B et D, suivant un résultat classique de Dipper–Mathas en type A pour les algèbres d'Ariki–Koike. Ainsi, la théorie des représentations des algèbres de Hecke affines de type B et D se réduit à l'étude de leurs quotients cyclotomiques où les valeurs propres sont dans une unique orbite pour la multiplication par  $q^2$  et l'inversion. La preuve consiste notamment en un théorème de décomposition pour des généralisations d'algèbres de Hecke carquois introduites récemment dans l'étude des algèbres de Hecke affines de type B et D, ramenant la situation générale d'un carquois non connexe avec involution à un cadre plus simple. Pour traiter simultanément les deux types, nous unifions les différentes définitions d'algèbres de Hecke carquois pour le type B déjà existantes.

## 1. Introduction

Cyclotomic quotients of the affine Hecke algebra of type A, also known as Ariki–Koike algebras, have been extensively studied since their introduction by Broué–Malle [5] and Ariki–Koike [2]. Given a field  $K$ , a subset  $I \subseteq K^\times$ , an element  $q \in K^\times$  and a finitely-supported family  $\Lambda = (\Lambda_i)_{i \in I}$  of non-negative integers, the Ariki–Koike algebra  $H^\Lambda(\mathfrak{S}_n)$  is defined by the generators  $g_0, \dots, g_{n-1}$  and the relations

$$\begin{aligned} g_i g_j &= g_j g_i, & \text{for all } i, j \in \{0, \dots, n-1\}, |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, & \text{for all } i \in \{1, \dots, n-2\}, \\ g_0 g_1 g_0 g_1 &= g_1 g_0 g_1 g_0, \\ (g_i - q)(g_i + q^{-1}) &= 0, & \text{for all } i \in \{1, \dots, n-1\}, \\ \prod_{i \in I} (g_0 - i)^{\Lambda_i} &= 0. \end{aligned}$$

We note that Ariki–Koike algebras are quotients, by the last relation, of affine Hecke algebras of type A and that the study of their representations (for all choices of  $I$  and  $\Lambda$ ) is equivalent to the study of finite-dimensional representations of affine Hecke algebras of type A.

By an important theorem of Dipper–Mathas [8], we know that it suffices to study Ariki–Koike algebras when the set  $I$  is  $q^2$ -connected, that is, in a single  $q^2$ -orbit (and even up to a scalar renormalisation of the generator  $g_0$ , when  $I \subseteq \langle q^2 \rangle$ ). More precisely, if  $I = \coprod_{j=1}^d I^{(j)}$  is the decomposition of  $I$  into  $q^2$ -connected sets, then we have a Morita equivalence

$$(♣) \quad H^\Lambda(\mathfrak{S}_n) \xrightarrow{\text{Morita}} \bigoplus_{\substack{n_1, \dots, n_d \geq 0 \\ n_1 + \dots + n_d = n}} \bigotimes_{j=1}^d H^{\Lambda^{(j)}}(\mathfrak{S}_{n_j}),$$

where  $\Lambda^{(j)}$  is the restriction of  $\Lambda$  to  $I^{(j)}$ . (Note that the assumption in [8] is slightly stronger than the one above, but in practice, it is this condition of  $q^2$ -connected sets that is used.) Hence, this Morita equivalence allows us to use results that are only known when the set  $I$  is  $q^2$ -connected, in particular, the celebrated Ariki's categorification theorem [1] that computes the decomposition numbers of Ariki–Koike algebras in terms of the canonical basis of a certain highest weight module over an affine quantum group.

Another way to obtain this Morita equivalence was given by the second author [21, §3.4], using the theory of quiver Hecke algebras. This is a family of graded algebras that was introduced independently by Khovanov–Lauda [15, 16] and Rouquier [22] a few years ago, in the context of categorification of quantum groups. If  $\Gamma$  is a quiver, we denote by  $R_n(\Gamma)$  the associated quiver Hecke algebra (see §3.1). For a certain quiver  $\Gamma$  depending only on the order of  $q^2$ , Brundan–Kleshchev [6] and independently Rouquier [22] proved that a certain “cyclotomic” quotient of  $R_n(\Gamma)$  is isomorphic to an Ariki–Koike algebra. This result is now a basic tool in the study of Ariki–Koike algebras and their degenerations, including the symmetric group and the classical Hecke algebra of type A. For instance, as consequences, first, the Ariki–Koike algebra inherits the grading of the cyclotomic quiver Hecke algebra and, second, it depends on  $q$  only through its order in  $K^\times$ . Now, if  $\Gamma$  is of the form  $\Gamma = \coprod_{j=1}^d \Gamma^{(j)}$ , where each  $\Gamma^{(j)}$  is a full subquiver, it was shown in [20, §6] that we have a decomposition

$$(\spadesuit) \quad R_n(\Gamma) \simeq \bigoplus_{\substack{n_1, \dots, n_d \geq 0 \\ n_1 + \dots + n_d = n}} \mathrm{Mat}_{\binom{n}{n_1, \dots, n_d}} \left( \bigotimes_{j=1}^d R_{n_j}(\Gamma^{(j)}) \right).$$

This isomorphism of algebras is compatible with cyclotomic quotients and combining with the previous isomorphism of Brundan–Kleshchev and Rouquier allows to recover the Morita equivalence  $(\clubsuit)$ . This Morita equivalence has been further generalised for the cyclotomic Hecke algebras of type  $G(r, p, n)$  [11]. We also indicate the paper [12], where the Dipper–Mathas result is studied and derived again from the point of view of affine Hecke algebras, and where the question of a similar result for other affine Hecke algebras is evoked.

The main point of this paper is to prove a similar decomposition theorem for some generalisations of quiver Hecke algebras and, hence, obtain an analogue of the Dipper–Mathas Morita equivalence for cyclotomic quotients of affine Hecke algebras of types B and D. Such generalisations of quiver Hecke algebras were introduced by Varagnolo and Vasserot [24] (for type B) and together with Shan [23] (for type D), in the course of their proofs of conjectures by Kashiwara–Enomoto [9] and Kashiwara–Miemietz [14]. For certain subcategories of representations of affine Hecke algebras of types B and D, these algebras play a similar role to quiver Hecke algebras for affine Hecke algebras of