Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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Tome 149
Fascicule 1

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 119-153

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 149, mars 2021

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
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ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
149 (1), 2021, p. 119-153

CHARACTERIZATION OF THE TWO-DIMENSIONAL FIVEFOLD TRANSLATIVE TILES

BY QI YANG & CHUANMING ZONG

ABSTRACT. — In 1885, Fedorov discovered that a convex domain can form a lattice tiling of the Euclidean plane, if and only if it is a parallelogram or a centrally symmetric hexagon. This paper proves the following results. Besides parallelograms and centrally symmetric hexagons, there is no other convex domain that can form any two, three or fourfold translative tiling in the Euclidean plane. In particular, it characterizes all two-dimensional fivefold translative tiles, which are parallelograms, centrally symmetric hexagons, two classes of octagons and one class of decagons.

RÉSUMÉ (*Caractérisation des pavages translatifs quintuples à deux dimensions*). — En 1885, Fedorov découvrait qu'un domaine convexe peut former un réseau-pavage de la plane euclidienne si et seulement s'il est un parallélogramme ou un hexagone symmétrique centralement. Cet article démontre les résultats suivants: outre les parallélogrammes et les hexagones symmétriques centralement, il n'y aucun autre domaine convexe qui peut former dans la plane eucldienne un pavage translatif double ou triple ou quadruple. En particulier, il caractérise tous les pavages translatifs quintuples en deux dimensions, qui sont parallélogrammes, hexagones symmétriques centralement, deux classes d'octogones, et une classe de décagons.

Texte reçu le 12 avril 2019, modifié le 5 octobre 2020, accepté le 20 octobre 2020.

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Mathematical subject classification (2010). — 52C20, 05B45, 51M20, 52C15.

Key words and phrases. — Multiple tiling, Translative tiling, Lattice tiling.

This work is supported by the National Natural Science Foundation of China (NSFC11921001), the National Key Research and Development Program of China (2018YFA0704701) and 973 Program 2013CB834201.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2021/119/$ 5.00
doi:10.24033/bsmf.2826

1. Introduction

In 1885, Fedorov [6] proved that *a convex domain can form a lattice tiling in the plane if and only if it is a parallelogram or a centrally symmetric hexagon; a convex body can form a lattice tiling in the space if and only if it is a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated dodecahedron, or a truncated octahedron.* As a generalized inverse problem of Fedorov's discovery, in 1900 Hilbert [13] listed the following question in the second part of his 18th problem: *Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.* To verify Hilbert's problem in the plane, in 1917 Bieberbach suggested to Reinhardt (see [19]) to determine all the two-dimensional convex tiles. However, to complete the list turns out to be challenging and dramatic. Over the years, the list has been successively extended by Reinhardt, Kershner, James, Rice, Stein, Mann, McLoud-Mann and Von Derau (see [15, 27]); its completeness has been mistakenly announced several times! In 2017, M. Rao [18] announced a completeness proof based on computer checks.

Let K be a convex body with (relative) interior $\mathrm{int}(K)$ and (relative) boundary $\partial(K)$, and let X be a discrete set, both in $\mathbb{E}^n$. We call $K+X$ a *translative tiling* of $\mathbb{E}^n$ and call K a *translative tile*, if $K+X=\mathbb{E}^n$ and the translates $\mathrm{int}(K)+\mathbf{x}_i$ are pairwise disjoint. In other words, if $K+X$ is both a packing and a covering in $\mathbb{E}^n$. In particular, we call $K+\Lambda$ a *lattice tiling* of $\mathbb{E}^n$ and call K a *lattice tile*, if Λ is an n-dimensional lattice. It is apparent that a translative tile must be a convex polytope. Usually, a lattice tile is called a *parallelohedron*.

As one can predict, to determine the parallelohedra in higher dimensions is complicated. According to Fedorov [6], there are exact five types of parallelohedra in $\mathbb{E}^3$. Through the works of Delone [3], Štogrin [23] and Engel [5], we know that there are exact 52 combinatorially different types of parallelohedra in $\mathbb{E}^4$. A computer classification for the five-dimensional parallelohedra was announced by Dutour Sikirić, Garber, Schürmann and Waldmann [4] only in 2015.

Let Λ be an n-dimensional lattice. The *Dirichlet–Voronoi cell* of Λ is defined by

$$C=\{\mathbf{x}:\mathbf{x}\in\mathbb{E}^n,\ |\mathbf{x},\mathbf{o}|\le|\mathbf{x},\Lambda|\}\,,$$

where $|X,Y|$ denotes the Euclidean distance between X and Y. Clearly, $C+\Lambda$ is a lattice tiling, and the Dirichlet–Voronoi cell C is a parallelohedron. In 1908, Voronoi [22] made a conjecture that *every parallelohedron is a linear transformation image of the Dirichlet–Voronoi cell of a suitable lattice.* In $\mathbb{E}^2$, $\mathbb{E}^3$ and $\mathbb{E}^4$, this conjecture was confirmed by Delone [3] in 1929. In higher dimensions, it is still open.

To characterize the translative tiles is another fascinating problem. At the first glance, translative tilings should be more complicated than lattice tilings. However, the dramatic story had a happy ending! It was shown by Minkowski [17] in 1897 that *every translative tile must be centrally symmetric*. In 1954, Venkov [21] proved that *every translative tile must be a lattice tile* (*parallelohedron*) (see [1] for generalizations). Later, a new proof for this beautiful result was independently discovered by McMullen [16].

Let X be a discrete multiset in $\mathbb{E}^n$ and let k be a positive integer. We call $K+X$ a *k-fold translative tiling* of $\mathbb{E}^n$ and call K a *translative k-tile*, if every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at least k translates of K in $K+X$, and every point $\mathbf{x} \in \mathbb{E}^n$ belongs to at most k translates of $\mathrm{int}(K)$ in $\mathrm{int}(K)+X$. In other words, $K+X$ is both a k-fold packing and a k-fold covering in $\mathbb{E}^n$ (see [7, 27]). In particular, we call $K+\Lambda$ a k-fold lattice tiling of $\mathbb{E}^n$ and call K a *lattice k-tile*, if Λ is an n-dimensional lattice. Apparently, a translative k-tile must be a convex polytope. In fact, similarly to Minkowski's characterization, it was shown by Gravin, Robins and Shiryaev [10] that *a translative k-tile must be a centrally symmetric polytope with centrally symmetric facets.*

Multiple tilings were first investigated by Furtwängler [8] in 1936 as a generalization of Minkowski's conjecture on cube tilings. Let C denote the n-dimensional unit cube. Furtwängler made a conjecture that *every k-fold lattice tiling $C+\Lambda$ has twin cubes. In other words, every multiple lattice tiling $C+\Lambda$ has two cubes sharing a whole facet.* In the same paper, he proved the two and three-dimensional cases. Unfortunately, when $n \geq 4$, this beautiful conjecture was disproved by Hajós [12] in 1941. In 1979, Robinson [20] determined all the integer pairs $\{n,k\}$ for which Furtwängler's conjecture is false. We refer to Zong [25, 26] for detailed accounts on this fascinating problem and to pages 82–84 of Gruber and Lekkerkerker [11] for some generalizations.

Let P denote an n-dimensional centrally symmetric convex polytope, let $\tau(P)$ be the smallest integer k, such that P can form a k-fold translative tiling in $\mathbb{E}^n$, and let $\tau^*(P)$ be the smallest integer k, such that P can form a k-fold lattice tiling in $\mathbb{E}^n$. For convenience, we define $\tau(P)=\infty$, if P cannot form translative tiling of any multiplicity. Clearly, for every centrally symmetric convex polytope, we have

$$\tau(P) \leq \tau^*(P).$$

In 1994, Bolle [2] proved that *every centrally symmetric lattice polygon is a lattice multiple tile.* However, little is known about the multiplicity. Let Λ denote the two-dimensional integer lattice and let P_8 denote the octagon with vertices $(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$, $(\frac{3}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{3}{2})$, $(-\frac{1}{2},-\frac{3}{2})$, $(-\frac{3}{2},-\frac{1}{2})$, $(-\frac{3}{2},\frac{1}{2})$ and $(-\frac{1}{2},\frac{3}{2})$, as shown in Figure 1. As a particular example of Bolle's theorem, it was discovered by Gravin, Robins and Shiryaev [10] that *$P_8+\Lambda$ is a sevenfold lattice tiling of $\mathbb{E}^2$.*