Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON CARLOTTO–SCHOEN-TYPE SCALAR-CURVATURE GLUINGS

Piotr T. Chruściel & Erwann Delay

Tome 149
Fascicule 4

2021

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 641-662

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 4, tome 149, décembre 2021

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
commandes@smf.emath.fr

AMS
P.O. Box 6248
Providence RI 02940
USA
www.ams.org

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
bulletin@smf.emath.fr • smf.emath.fr

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
149 (4), 2021, p. 641-662

ON CARLOTTO–SCHOEN-TYPE SCALAR-CURVATURE GLUINGS

by Piotr T. Chruściel & Erwann Delay

Abstract. — We carry out a Carlotto–Schoen-type gluing with interpolating scalar curvature on cone-like sets, or deformations thereof, in the category of smooth Riemannian asymptotically Euclidean metrics.

Résumé (*Sur les recollements de courbure scalaire de type Carlotto-Schoen*). — Nous effectuons des recollements de type Carlotto-Schoen en interpolant la courbure scalaire sur des ensembles de type cône ou des déformations de ceux-ci, dans la catégorie des métriques Riemanniennes lisses asymptotiquement Euclidiennes.

1. Introduction

In an outstanding paper [6], Carlotto and Schoen constructed non-trivial asymptotically Euclidean scalar-flat metrics that are Minkowskian outside of a

Texte reçu le 5 mai 2019, modifié le 24 septembre 2021, accepté le 7 octobre 2021.

Piotr T. Chruściel, University of Vienna, Austria • *E-mail :* piotr.chrusciel@univie.ac.at • *Url :* http://homepage.univie.ac.at/piotr.chrusciel/
Erwann Delay, Avignon Université, Laboratoire de Mathématiques d'Avignon, F-84916 Avignon ; Aix Marseille Université, F.R.U.M.A.M. – CNRS, F-13331 Marseille • *E-mail :* Erwann.Delay@univ-avignon.fr • *Url :* https://erwanndelay.wordpress.com

Mathematical subject classification (2010). — 53C21, 58J99, 35J30.

Key words and phrases. — Scalar curvature, gluing, asymptotically Euclidean.

This work was supported in part by the Austrian Research Fund (FWF), Project P 29517-N16, and by the grant ANR-17-CE40-0034 of the French National Research Agency ANR (project CCEM).

0037-9484/2021/641/$ 5.00
doi:10.24033/bsmf.2840

solid cone. Their paper strongly suggests, although no explicit statements are made, that their construction can be generalised as follows:

1. Rather than gluing an asymptotically Euclidean metric to a flat one, any two asymptotically Euclidean metrics g_1 and g_2 can be glued together.
2. In the spirit of [12], the gluings at zero-scalar curvature can be replaced by gluings where the scalar curvature of the final metric equals
$$\chi R(g_1) + (1-\chi) R(g_2)$$
with a cut-off function χ.
3. The geometry of the gluing region can be somewhat more general than the interface between two rotationally symmetric cones.

The aim of this paper is to give detailed proofs of the above. While the overall structure of the argument is rather similar, the details are different. Both the proof in [6] and ours require a weighted Poincaré inequality, with nonstandard weights adapted to the geometry of the problem at hand. We provide a version of this inequality that is different from the one in [6], which might be of more general interest. We also provide a detailed proof of the gluing with exponential weights near the boundary, which is only sketched in [6]. Finally, we replace some of the tailor-made arguments in [6] by off-the-shelf results from [8].

A special case of our main Theorem 3.1 below provides the following variant of the Riemannian-geometry version of the main theorem of [6]:

Theorem 1.1. — *Let $n \geq 3$. Consider two smooth n-dimensional asymptotically Euclidean Riemannian metrics g_1 and g_2 and two nested cones with non-zero opening angles and vertices displaced along the symmetry axis, see Figure 1.1. Simultaneously scaling-up the cones if necessary, or simultaneously shifting them to the asymptotic region, there exists a smooth asymptotically Euclidean metric g, which coincides with g_1 outside of the larger cone and with g_2 inside the smaller one, with the scalar curvature $R(g)$ lying between $R(g_1)$ and $R(g_2)$ in the intermediate region.*

In particular, if $R(g_1) = R(g_2) = 0$, then $R(g) = 0$.

In Section 6, we show how Theorem 1.1 follows from our main Theorem 3.1 below. The key to the proof of this last theorem is a weighted Poincaré inequality involving radially scaled exponential weights, proved in Proposition 5.6 below. The rest of the proof is a verification of the hypotheses of the rather general results proved in [8].

As was already mentioned, Theorem 3.1 below allows more general cones than the ones in [6], as described at the beginning of Section 3. In fact, our arguments apply to a large class of deformed cones as well, e.g. "logarithmically rotated ones"; cf. Theorem 4.1 and the discussion in Section 4.

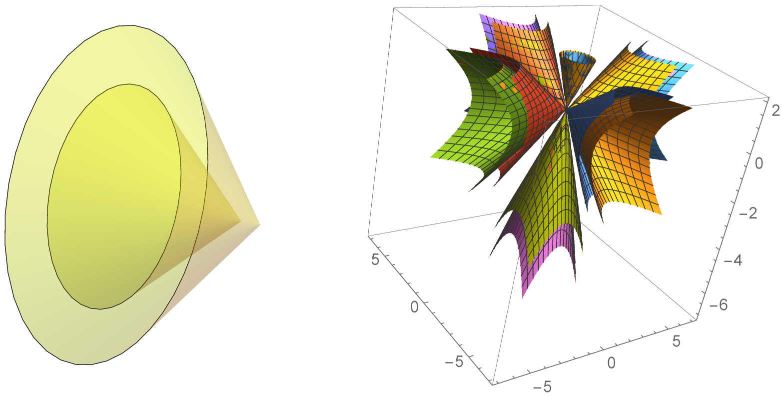

FIGURE 1.1. Left: If the second metric is flat and both metrics are scalar-flat, then the new metric is flat inside the smaller cone, scalar-flat everywhere, and coincides with the first one outside the larger cone. Both cones extend to infinity, and their tips are located very far in the asymptotically Euclidean region. Right: Iterating the construction one can embed any finite number of distinct initial data sets into Minkowskian data, or paste-in Minkowskian data inside several cones into a given data set.

2. Definitions, notations and conventions

We use the summation convention throughout; indices are lowered with g_{ij} and raised with its inverse g^{ij}.

We will have to control the asymptotic behaviour of the objects at hand frequently. Given a tensor field T and a function f, we will write

$$T = O(f),$$

when there exists a constant C such that the g-norm of T is dominated by Cf.

A metric g on a manifold M will be said to be asymptotically Euclidean (AE), if M contains a region, diffeomorphic to the complement of a ball in $\mathbb{R}^n$, on which the metric g approaches the Euclidean metric δ as one recedes to infinity.

Let ϕ and ψ be two smooth strictly positive functions on an n-dimensional manifold M. The function ψ will be used to control the growth of the fields involved near boundaries or in the asymptotic regions, while ϕ will control how the growth is affected by derivations. For $k \in \mathbb{N}$, let $H^k_{\phi,\psi}(g)$ be the space of