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IN A FAMILY OF TEICHMÜLLER DISKS

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ALGEBRAIC INTERSECTION FOR TRANSLATION SURFACES IN A FAMILY OF TEICHMÜLLER DISKS

BY SMAIL CHEBOUI, AREZKI KESSI & DANIEL MASSART

ABSTRACT. — This paper deals with the question: on a translation surface, naturally endowed with a flat metric with conical singularities, how much can two curves of a given length intersect? This leads to the quantity $KVol$ defined as the supremum, over all pairs of closed geodesics, of the ratio of their algebraic intersection to the product of their length, times the total area of the surface (to make it scaling-invariant). We give a hyperbolic-geometric construction to compute the quantity $KVol$ in a family of Teichmüller disks of square-tiled surfaces with stair-shaped templates.

RÉSUMÉ (Intersection algébrique pour une famille de disques de Teichmüller de surfaces de translation). — Cet article traite de la question suivante : pour une surface de translation, naturellement munie d'une métrique plate à singularités coniques, combien de fois deux courbes d'une longueur donnée peuvent-elles s'intersecter ? On est amené à considérer la quantité $KVol$ définie comme le supremum, sur toutes les paires de géodésiques fermées, du rapport entre leur intersection algébrique et le produit de leurs longueurs, fois le volume total de la surface (afin d'avoir une quantité invariante par dilatation). On donne une construction, en géométrie hyperbolique, permettant de calculer $KVol$ pour une famille de disques de Teichmüller de surfaces de translation obtenues à partir de polygones en escalier.

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1. Introduction

1.1. Definitions. — Let X be a closed surface, that is, a compact, connected manifold of dimension 2, without boundary. Let us assume that X is oriented. If two C^1 closed curves α and β in X intersect transversally at a point $P \in X$, we set $\text{Int}_P(\alpha, \beta) = 1$, if β crosses α from right to left, and $\text{Int}_P(\alpha, \beta) = -1$ otherwise. Then the algebraic intersection $\text{Int}(\alpha, \beta)$ of α and β is the sum over all intersection points P of $\text{Int}_P(\alpha, \beta)$. The algebraic intersection endows the first homology $H_1(X, \mathbb{R})$ with a symplectic bilinear form. In particular, $\text{Int}(\alpha, \beta)$ is finite and only depends on the homology classes of α and β .

Now let us assume that X is endowed with a Riemannian metric g . We denote $\text{Vol}(X, g)$ the Riemannian volume of X with respect to the metric g , and for any piecewise smooth closed curve α in X , we denote $l_g(\alpha)$ the length of α with respect to g . When there is no ambiguity, we omit the reference to g .

We are interested in the quantity

$$(1) \quad \text{KVol}(X, g) = \text{Vol}(X, g) \sup_{\alpha, \beta} \frac{\text{Int}(\alpha, \beta)}{l_g(\alpha) l_g(\beta)},$$

where the supremum ranges over all piecewise smooth closed curves α and β in X . The $\text{Vol}(X, g)$ factor is there to make KVol invariant to rescaling of the metric g . See [9] as to why KVol is finite. The quantity KVol comes up naturally when you want to compare the stable norm (a norm that measures the length of a homology class, with respect to the metric g) with the Hodge norm (or L^2 -norm) on $H_1(M, \mathbb{R})$ (see [9]). It is easy to change the metric to make KVol go to infinity, you just need to pinch a nonseparating closed curve α to make its length go to zero. The interesting surfaces are those (X, g) for which KVol is small.

Another motivation for studying KVol is the analogy with the systolic volume. Recall that the systole of a Riemannian manifold is the shortest length of a noncontractible, closed geodesic, and the systolic volume is the ratio of the total area to the square of the systole. As with KVol , the interesting metrics are those for which the systolic volume is small. In [9], some inequalities relating KVol to the systolic volume are given. Minimizing metrics for the systolic volume are famously hard to find but are known in some special cases. In this paper, we look for minimizers of KVol , also in a special case. Note that while the systolic volume is a maximum if X is compact, KVol is only a supremum, which may, or may not, be a maximum.

When X is the torus, we have $\text{KVol}(X, g) \geq 1$, with equality if and only if the metric g is flat (see [9]). Furthermore, when g is flat, the supremum in (1) is not attained, except for a negligible subset of the set of all flat metrics. In [9], KVol is studied as a function of g , on the moduli space of hyperbolic (that is, the curvature of g is -1) surfaces of fixed genus. It is proved that KVol goes to infinity when g degenerates by pinching a nonseparating closed curve, while

KVol remains bounded when g degenerates by pinching a separating closed curve.

This leaves open the question of whether KVol has a minimum over the moduli space of hyperbolic surfaces of genus s , for $s \geq 2$. It is conjectured in [9] that for almost every (X, g) in the moduli space of hyperbolic surfaces of genus s , the supremum in (1) is attained (that is, it is actually a maximum).

In this paper, we consider a different class of surfaces: translation surfaces of genus s , with one conical point. For our purposes, a translation surface is a plane polygon, with parallel sides of equal length pairwise identified, in such a way that the quotient space is an orientable two-dimensional manifold (see Figure 1.1). A translation surface comes equipped with a flat metric, possibly singular at the images, after identification, of the vertices of the polygon, since the total angle at the image of a vertex may be $2k\pi$, with $k > 1$. A point with total angle $> 2\pi$ is called a singularity or conical point. For instance, in Figure 1.1, there is only one singularity, with a total angle $2(2s - 1)\pi$. Since we can measure lengths, it makes sense to compute KVol for a translation surface.

More precisely, a translation surface is an equivalence class of polygons with parallel sides of equal length pairwise identified, where polygons P_1 and P_2 are said to be equivalent if P_1 may be cut into polygonal pieces and the pieces rearranged by translations, into P_2 (see [8], Figures 12–15, some of which are taken from [2]). This equivalence relation respects the number of singular points and the total angle at any given singular point. Two equivalent polygons yield isometric surfaces. Moreover, if we apply a similitude (rotation + rescaling) to some polygon, in general, we get a different translation surface, but KVol does not change.

The set (called stratum in the literature) of translation surfaces with one conical point of angle $2(2s - 1)\pi$ is denoted $\mathcal{H}(2s - 2)$ (see [6]).

We consider the family of translation surfaces $St(2s - 1)$ (so named after [11]) depicted in Figure 1.1, for $s \in \mathbb{N}, s \geq 2$, obtained by gluing the opposite sides of a staircase-shaped template made of $2s - 1$ squares (see Figure 1.1).

Our first result is (see Corollary 3.3)

$$\forall s \geq 2, \text{KVol}(St(2s - 1)) = 2s - 1.$$

This is the first exact computation of KVol , outside of flat tori.

Before stating our next result we need to elaborate a little bit on the Teichmüller disk of $St(2s - 1)$.

1.2. The Teichmüller disc \mathcal{T} of $St(2s - 1)$. — Let us explain the terminology. Every translation surface may be viewed as a plane template with parallel sides of equal length pairwise identified. The group $GL_2^+(\mathbb{R})$ acts linearly on templates, preserving identifications, so it acts on translation surfaces. It may happen that for some template T and some element A of $GL_2^+(\mathbb{R})$, both T and $A.T$ are templates of the same translation surface X . Then we say that A lies