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DYNAMICS OF BUBBLING WAVE MAPS WITH PRESCRIBED RADIATION

BY Jacek JENDREJ, Andrew LAWRIE AND Casey RODRIGUEZ

Abstract. – We study energy critical one-equivariant wave maps taking values in the two-sphere. It is known that any finite energy wave map that develops a singularity does so by concentrating the energy of (possibly) several copies of the ground state harmonic map at the origin. If only a single bubble of energy is concentrated, the solution decomposes into a dynamically rescaled harmonic map plus a term that accounts for the energy that radiates away from the singularity. In this paper, we construct blow up solutions by prescribing the radiative component of the map. In addition, we give a sharp classification of the dynamical blow up rate for every solution with this prescribed radiation.

Résumé. – Nous étudions l'équation des applications d'onde (wave maps) critique pour l'énergie, à valeurs dans la sphère de dimension 2, dans le cas équivariant de degré 1. Il a été montré qu'une application d'onde d'énergie finie ne peut développer de singularité qu'en concentrant à l'origine du système des coordonnées des bulles d'énergie, c'est-à-dire des applications harmoniques remises à l'échelle. S'il n'y a qu'une seule bulle, alors l'application se décompose en la superposition de celle-là, et de la radiation émanant de la singularité. Dans cet article, nous construisons des solutions explosives en prescrivant la composante radiative de l'application. Nous déterminons également le taux de concentration de la bulle explosive pour toute solution ayant la même radiation que la solution construite.

1. Introduction

We consider wave maps from (1+2)-dimensional Minkowski space $\mathbb{R}^{1+2}_{t,x}$ to the 2-sphere, $\mathbb{S}^2$, with 1-equivariant symmetry. In this setting, the objects under consideration are formal critical points of the Lagrangian action,

$$\mathcal{A}(U) = \frac{1}{2}\int_{\mathbb{R}^{1+2}_{t,x}} \Big(-|\partial_t U(t,x)|^2 + |\nabla U(t,x)|^2 \Big)\, \mathrm{d}x\, \mathrm{d}t, \tag{1.1}$$

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where $U : \mathbb{R}^{1+2}_{t,x} \to \mathbb{S}^2 \subset \mathbb{R}^3$ are maps that take the restricted form,

$$U(t,r,\theta) = (\sin u(t,r)\cos\theta, \sin u(t,r)\sin\theta, \cos u(t,r)) \in \mathbb{S}^2 \subset \mathbb{R}^3. \tag{1.2}$$

Here (r,θ) are polar coordinates on $\mathbb{R}^2$, and $u(t,r)$ is a radially symmetric function. The Cauchy problem for 1-equivariant wave maps reduces to a scalar semilinear wave equation for the polar angle u:

$$\begin{aligned} &\partial_t^2 u - \partial_r^2 u - \frac{1}{r}\partial_r u + \frac{\sin 2u}{2r^2} = 0, \quad (t,r) \in \mathbb{R}\times(0,\infty), \\ &(u(t_0), \partial_t u(t_0)) = (u_0, \dot u_0), \qquad t_0 \in \mathbb{R}. \end{aligned} \tag{1.3}$$

Wave maps are referred to as nonlinear σ-models in the high energy physics literature, see e.g., [20, 9]. From the mathematical point of view, they are a canonical example of a geometric wave equation as they simultaneously generalize the free scalar wave equation to manifold valued maps and the classical harmonic maps equation to Lorentzian domains. The case considered here is of particular interest, as the static solutions given by finite energy harmonic maps are amongst the simplest examples of topological solitons; other examples include kinks in scalar field equations, vortices in Ginzburg-Landau equations, Dirac monopoles, Skyrmions, and Yang-Mills instantons; see [20]. The symmetry reduced Equation (1.3) is a much studied problem, since it admits intriguing features from the point of view of dynamics, e.g., bubbling harmonic maps, multi-soliton solutions, etc., in the relatively simple setting of a geometrically natural semilinear wave equation. For a more thorough presentation of the physical or geometric content of this equation, see e.g., [20, 40, 9].

The energy functional $\mathcal{E}$ is defined for vectors $\boldsymbol{u}_0 := (u_0(\cdot), \dot u_0(\cdot))$ by the formula

$$\mathcal{E}(\boldsymbol{u}_0) := \pi\int_0^\infty \Big(|\dot u_0|^2 + |\partial_r u_0|^2 + \frac{\sin^2 u_0}{r^2}\Big)\, r\,\mathrm{d}r. \tag{1.4}$$

Due to the temporal translation invariance of the action $\mathcal{A}(\cdot)$ and Noether's theorem, the energy is conserved if u is a solution to (1.3): $\mathcal{E}(\boldsymbol{u}(t)) = \mathcal{E}(\boldsymbol{u}_0)$ for all t where $\boldsymbol{u}(t) := (u(t,\cdot), \partial_t u(t,\cdot))$. Note that initial data $\boldsymbol{u}_0 = (u_0, \dot u_0)$ of finite energy forces $u_0(r) \to m\pi$ as $r \to 0$ and $u_0(r) \to n\pi$ as $r \to \infty$ for $m, n \in \mathbb{Z}$. We will fix $m = 0$, and $n = 1$ in our analysis, but we could just as well consider states of finite energy with arbitrary endpoint in $\pi\mathbb{Z}$. Thus the finite energy maps we study connect the north and south poles of $\mathbb{S}^2$ and have topological degree one, i.e., they are members of the space

$$\mathcal{H}_1 := \{\boldsymbol{u}_0 \mid \mathcal{E}(\boldsymbol{u}_0) < \infty, \lim_{r\to 0} u_0(r) = 0, \quad \lim_{r\to\infty} u_0(r) = \pi\}. \tag{1.5}$$

The family of stationary solutions

$$Q_\lambda(r) := 2\arctan\big(\frac{r}{\lambda}\big), \quad \lambda > 0 \tag{1.6}$$

plays a fundamental role in the study of (1.3). We will write $Q(r) := Q_1(r)$. We note that $\boldsymbol{Q} := (Q, 0) \in \mathcal{H}_1$ since $Q(r) \to \pi$ as $r \to \infty$. In fact, Q, which is the polar angle of a degree one *harmonic map*, has minimal energy in $\mathcal{H}_1$; see (2.13) below.

We will often work with vectors in the energy space $\mathcal{H} := H \times L^2$, where the space H is the completion of $C_0^\infty((0,\infty))$ for the norm

$$\|v\|_H^2 := \pi\int_0^\infty \Big(|\partial_r v(r)|^2 + \frac{|v(r)|^2}{r^2}\Big)\, r\,\mathrm{d}r. \tag{1.7}$$

Note that $v \in H$ forces $\lim_{r\to+\infty} v(r) = 0$ and that $\boldsymbol{u}_0 \in \mathcal{H}_1$ means that $u_0 - Q_\lambda \in H$ for any fixed $\lambda > 0$. It is classical that (1.3) is locally well-posed for initial data in $\mathcal{H}$; see [41] (and thus, (1.8) is well posed in $\mathcal{H}$).

We observe that the wave maps Equation (1.3) can be equivalently expressed as a first order system:

$$\begin{aligned} \partial_t \boldsymbol{u}(t) &= J \circ D\mathcal{E}(\boldsymbol{u}(t)) \\ \boldsymbol{u}(t_0) &= \boldsymbol{u}_0, \end{aligned} \tag{1.8}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D\mathcal{E}(\boldsymbol{u}(t)) = \begin{pmatrix} -\partial_r^2 u(t) - \frac{1}{r}\partial_r u(t) + \frac{\sin 2u(t)}{2r^2} \\ \partial_t u(t) \end{pmatrix}. \tag{1.9}$$

1.1. Main results

The breakthrough works of Krieger, Schlag, Tataru [18, 19], Rodnianski, Sterbenz [38], and Raphaël, Rodnianski [36] proved that wave maps can develop singularities in finite time by concentrating energy at a point in space. The main goal of this paper is to directly tie the precise blow-up dynamics of concentrating wave maps to the part of the solution that radiates away from the singularity.

We start by clarifying what is meant by the *radiation field* of a singular wave map. By equivariance and energy-criticality a solution to (1.3) can only become singular by concentrating energy at $r = 0$. In [43], Struwe proved that such energy concentration can only occur via the bubbling of at least one harmonic map, at least along a sequence of times. It was later shown in [6, Theorem 1.3] that any 1-equivariant wave map $u(t) \in \mathcal{H}_1$ with energy $\mathcal{E}(\boldsymbol{u}) < 3\mathcal{E}(\boldsymbol{Q})$ that blows up, say as time $t \to 0^+$, admits a decomposition of the form

$$\begin{aligned} &\boldsymbol{u}(t) = \boldsymbol{Q}_{\lambda(t)} + \boldsymbol{u}_0^* + \boldsymbol{g}(t), \\ &\lambda(t) = o(t) \ \text{ as } \ t \to 0^+, \\ &\|\boldsymbol{g}(t)\|_{\mathcal{H}} \to 0 \ \text{ as } \ t \to 0^+, \end{aligned} \tag{1.10}$$

where $\boldsymbol{u}_0^* \in \mathcal{H}$ is uniquely determined by $\boldsymbol{u}(t) = (u(t), \partial_t u(t))$, and $\lambda(t)$ is a continuous function. In fact, letting $\boldsymbol{u}^*(t) = (u^*(t), \partial_t u^*(t))$ denote the wave map evolution of the initial data $\boldsymbol{u}^*(0) = \boldsymbol{u}_0^*$ (i.e., the solution to (1.8)), we have $\mathcal{E}(\boldsymbol{u}^*) = \mathcal{E}(\boldsymbol{u}) - \mathcal{E}(\boldsymbol{Q})$ and,

$$\boldsymbol{u}(t, r) = (\pi, 0) + \boldsymbol{u}^*(t, r), \quad \forall r \geq t, \tag{1.11}$$

i.e., $\boldsymbol{u}^*(t)$ accounts for the part of $\boldsymbol{u}(t)$ that radiates out of the light cone as we approach the singular time. We will refer to $\boldsymbol{u}_0^* \in \mathcal{H}$ or the associated flow $\boldsymbol{u}^*$ as the *radiation field* of the singular wave map $\boldsymbol{u}$.

In the case of a wave map that blows up by bubbling off a *single* harmonic map, the cap on the energy $\mathcal{E}(\boldsymbol{u}) < 3\mathcal{E}(\boldsymbol{Q})$ (which ensures that there can only be one blow up bubble) was removed by Côte [5], and Jia, Kenig [16] generalized the result to k-equivariant maps. These works also provided a further generalization of (1.10) that allows for the possibility of more concentrating bubbles along a sequence of times. Later, Duyckaerts, Jia, Kenig, and Merle [7] established a one-bubble decomposition as in (1.10) for general wave maps with energy slightly above the ground state harmonic map.