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HOMOTOPY INVARIANCE OF HIGHER SIGNATURES AND 3-MANIFOLD GROUPS

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To the memory of Michel Matthey

ABSTRACT. — For closed oriented manifolds, we establish oriented homotopy invariance of higher signatures that come from the fundamental group of a large class of orientable 3-manifolds, including the “piecewise geometric” ones in the sense of Thurston. In particular, this class, that will be carefully described, is the class of all orientable 3-manifolds if the Thurston Geometrization Conjecture is true. In fact, for this type of groups, we show that the Baum-Connes Conjecture With Coefficients holds. The non-oriented case is also discussed.

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Résumé (*Invariance homotopique des signatures supérieures et groupes fondamentaux des variétés de dimension* 3)

Nous démontrons que pour des variétés fermées et orientées les signatures qui proviennent des groupes fondamentaux d'une large classe de variétés orientables de dimension 3 sont des invariants homotopiques. Cette classe, que nous décrivons soigneusement, contient en particulier les variétés géométriques par morceaux au sens de Thurston. Si la conjecture de géométrisation de Thurston s'avère vraie cette classe coïncide alors avec celle des groupes fondamentaux de variétés de dimension 3 orientables. Plus précisément nous démontrons que tous les groupes dans cette classe satisfont la conjecture de Baum-Connes avec coefficients. Nous discutons également le cas non-orientable.

1. Introduction and statement of the main results

We assume all manifolds to be non-empty, pointed (*i.e.* we fix a base-point), second countable, Hausdorff and smooth. Given a closed connected oriented manifold M^m of dimension m, let $[M]$ denote either orientation classes in $H_m(M;\mathbb{Q})$ and in $H^m(M;\mathbb{Z})$, and let $\mathcal{L}_M \in H^{4*}(M;\mathbb{Q})$ be the Hirzebruch L-class of M, which is defined as a suitable rational polynomial in the Pontrjagin classes of M (see [23, pp. 11–12] or [37, Ex. III.11.15]). Denote the usual Kronecker pairing for M, with rational coefficients, by

$$\langle\,.\,,.\,\rangle : H^*(M;\mathbb{Q}) \times H_*(M;\mathbb{Q}) \longrightarrow \mathbb{Q}.$$

If M is of dimension $m = 4k$, then the Hirzebruch Signature Theorem (see [23, Thm. 8.2.2] or [37, p. 233]) says that the rational number $\langle \mathcal{L}_M, [M]\rangle$ is the signature of the cup product quadratic form

$$H^{2k}(M;\mathbb{Z}) \otimes H^{2k}(M;\mathbb{Z}) \longrightarrow H^{4k}(M;\mathbb{Z}) = \mathbb{Z}\cdot[M] \cong \mathbb{Z}, \quad (x,y) \longmapsto x \cup y.$$

As a consequence, $\langle \mathcal{L}_M, [M]\rangle$ is an integer and an oriented homotopy invariant of M (among closed connected oriented manifolds, hence of the same dimension $4k$). In 1965, S.P. Novikov proposed the following conjecture, now known as the *Novikov Conjecture* or as the *Novikov Higher Signature Conjecture*: Let G be a discrete group, let BG be its classifying space, and let $\alpha \in H^*(BG;\mathbb{Q}) \cong H^*(G;\mathbb{Q})$ be a prescribed rational cohomology class of BG. Now, for a closed connected oriented manifold M^m (with m arbitrary) and for a continuous map $f\colon M \longrightarrow BG$, consider the α-*higher signature* (*coming from* G)

$$\mathrm{sign}^G_\alpha(M,f) := \big\langle f^*(\alpha) \cup \mathcal{L}_M, [M]\big\rangle \ \in\ \mathbb{Q},$$

where $f^*\colon H^*(BG;\mathbb{Q}) \longrightarrow H^*(M;\mathbb{Q})$ is induced by f. Then, the conjecture predicts that the rational number $\mathrm{sign}_\alpha^G(M,f)$ is an oriented homotopy invariant of the pair (M,f), in the precise sense that $\mathrm{sign}_\alpha^G(N,g) = \mathrm{sign}_\alpha^G(M,f)$ whenever N^n is a second closed connected oriented manifold equipped with a continuous map $g\colon N \longrightarrow BG$, and such that there exists a homotopy equivalence $h\colon M \xrightarrow{\simeq} N$ preserving the orientation, that is, $h_*[M] = [N]$ in $H_m(N;\mathbb{Q})$ (automatically, $m = n$), and with $g \circ h \simeq f$, *i.e.* the diagram

$$\begin{tikzcd} M \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft\simeq"] \arrow[r, "f"] & BG \\ N \arrow[ur, "g"'] & \end{tikzcd}$$

commutes up to homotopy, as indicated. If, for a given group G, this holds for every rational cohomology class $\alpha \in H^*(BG;\mathbb{Q})$, then one says that G verifies the Novikov Conjecture. Of particular interest are the "*self higher signatures*" of a closed connected oriented manifold M, namely those corresponding to the case $G := \pi_1(M)$, for some chosen cohomology class $\alpha \in H^*(BG;\mathbb{Q})$, with, as map $f\colon M \longrightarrow BG$, 'the' classifying map of the universal covering space $\widetilde{M}$ of M (up to homotopy). Special attention is deserved by the situation where M is aspherical, in which case one can take M as a model for BG, and $f := \mathrm{id}_M$.

Now, fix a discrete group G (countable, say) and let $\mathbb{C}G$ be the complex group algebra of G. Then $\mathbb{C}G$ is equipped with an involution

$$\lambda_{g_1} g_1 + \cdots + \lambda_{g_k} g_k \longmapsto \bar{\lambda}_{g_1} g_1^{-1} + \cdots + \bar{\lambda}_{g_k} g_k^{-1}$$

and any unitary representation U of G on a Hilbert space H_U gives rise to an involutive representation π_U of $\mathbb{C}G$ on H_U. The maximal C^*-algebra of G, denoted by C^*G is then the completion of $\mathbb{C}G$ with respect to the norm

$$\|\bullet\|_{\max} := \sup_U \left\|\pi_U(\bullet)\right\|_{H_U},$$

where U runs through all unitary representations of G. On the other hand, the reduced C^*-algebra of G, denoted by $C^*_r G$, is by definition the completion of $\mathbb{C}G$ with respect to the norm

$$\|\bullet\|_r := \left\|\pi_\lambda(\bullet)\right\|_{\ell^2(G)},$$

where λ is the left regular representation, i.e the representation of G on $\ell^2(G)$ given by left translations. Notice that we have an obvious surjective map

$$\lambda^G\colon C^*G \relbar\joinrel\twoheadrightarrow C^*_r G.$$

Let $K_*(-)$ and $K_*^{\mathrm{top}}(-)$ denote respectively complex topological K-homology with compact supports for spaces and analytical K-theory for complex Banach algebras. In [42], Miščenko defines a group homomorphism

$$\tilde{\nu}_*^G\colon K_*(BG) \longrightarrow K_*^{\mathrm{top}}(C^*G)$$

and shows that if $\tilde{\nu}_*^G$ is rationally injective, *i.e.* injective after tensoring with $\mathbb{Q}$, then the Novikov Conjecture holds for G. The composite

$$\nu_*^G\colon K_*(BG) \xrightarrow{\tilde{\nu}_*^G} K_*^{\mathrm{top}}(C^*G) \xrightarrow{\lambda_*^G} K_*^{\mathrm{top}}(C_r^*G)$$

is called the Novikov assembly map and the so-called *Strong Novikov Conjecture* for G is the statement that ν_*^G is rationally injective, and this, again, implies the usual Novikov Conjecture. Next, we explain the connection with the Baum-Connes Conjecture. Let $\underline{EG}$ denote the universal example for proper actions of G (in other words, up to G-homotopy, the classifying space for the family of finite subgroups of G); by definition, this is a locally compact Hausdorff proper (left, say) G-space such that for any locally compact Hausdorff G-space X, there exists a G-map from X to $\underline{EG}$, and any two G-maps from X to $\underline{EG}$ are G-homotopic. For instance, the universal covering space $EG := \widetilde{BG}$ of BG is a model for $\underline{EG}$ when G is torsion-free; the point pt is a model for $\underline{EG}$ when G is finite; if G is a discrete subgroup of an almost connected Lie group Γ with maximal compact subgroup K, then Γ/K is a model for $\underline{EG}$. Suppose further given a separable G-C^*-algebra $\mathcal{A}$. Then, there is a suitable G-equivariant K-homology group $K_*^G(\underline{EG};\mathcal{A})$ and a specific group homomorphism, called the Baum-Connes assembly map with coefficients in $\mathcal{A}$,

$$\mu_*^{G,\mathcal{A}}\colon K_*^G(\underline{EG};\mathcal{A}) \longrightarrow K_*^{\mathrm{top}}(\mathcal{A} \rtimes_r G),$$

where $\mathcal{A} \rtimes_r G$ is the reduced C^*-crossed product of $\mathcal{A}$ by G. The group G is said to satisfy the *Baum-Connes Conjecture With Coefficients* if the assembly map $\mu_*^{G,\mathcal{A}}$ is an isomorphism for any separable G-C^*-algebra $\mathcal{A}$. If this is at least known to be fulfilled for the C^*-algebra $\mathbb{C}$ with trivial G-action, then one says that G verifies the *Baum-Connes Conjecture* (*i.e.* without mentioning coefficients). In this special case where $\mathcal{A} = \mathbb{C}$ with trivial G-action, one has $\mathcal{A} \rtimes_r G = C_r^*G$ and $K_*^G(\underline{EG};\mathcal{A}) = K_*^G(\underline{EG})$, the G-equivariant K-homology group with G-compact supports of $\underline{EG}$, and the corresponding assembly map boils down to a map

$$\mu_*^G := \mu_*^{G,\mathbb{C}}\colon K_*^G(\underline{EG}) \longrightarrow K_*^{\mathrm{top}}(C_r^*G).$$

This is linked with the Novikov Conjecture as follows. First, since G acts properly and freely on EG, and since $BG \simeq G\backslash EG$, there is a canonical isomorphism

$$K_*(BG) \cong K_*^G(EG).$$

Secondly, since tautologically any proper and free G-action is proper, there is a G-map $EG \longrightarrow \underline{EG}$, unique up to G-homotopy, and the induced map

$$K_*^G(EG) \longrightarrow K_*^G(\underline{EG})$$