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Fabiano G.B. Brito & Pablo M. Chacón & David L. Johnson

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UNIT VECTOR FIELDS ON ANTIPODALLY PUNCTURED SPHERES: BIG INDEX, BIG VOLUME

BY FABIANO G.B. BRITO, PABLO M. CHACÓN & DAVID L. JOHNSON

ABSTRACT. — We establish in this paper a lower bound for the volume of a unit vector field $\vec{v}$ defined on $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. This lower bound is related to the sum of the absolute values of the indices of $\vec{v}$ at x and $-x$.

RÉSUMÉ (*Champs unitaires dans les sphères antipodalement trouées : grand indice entraîne grand volume*)

Nous établissons une borne inférieure pour le volume d'un champ de vecteurs $\vec{v}$ défini dans $\mathbf{S}^n \setminus \{\pm x\}$, $n = 2, 3$. Cette borne inférieure dépend de la somme des valeurs absolues des indices de $\vec{v}$ en x et en $-x$.

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FABIANO G.B. BRITO, Dpto. de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, R. do Matão 1010, São Paulo-SP, 05508-090 (Brazil) • *E-mail :* `fabiano@ime.usp.br`

PABLO M. CHACÓN, Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca (Spain) • *E-mail :* `pmchacon@usal.es`

DAVID L. JOHNSON, Department of Mathematics, Lehigh University, 14 E. Packer Avenue, Bethlehem, PA, 18015 (USA) • *E-mail :* `david.johnson@lehigh.edu`

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1. Introduction

The volume of a unit vector field $\vec{v}$ on a closed Riemannian manifold M is defined [10] as the volume of the section $\vec{v} : M \to T^1M$, where the Sasakian metric is considered in T^1M. The volume of $\vec{v}$ can be computed from the Levi-Civita connection ∇ of M. If we denote by ν the volume form, for an orthonormal local frame $\{e_a\}_{a=1}^n$, we have

$$(1)\qquad \operatorname{vol}(\vec{v}) = \int_M \Big(1 + \sum_{a=1}^{n} \|\nabla_{e_a}\vec{v}\|^2 + \sum_{a_1<a_2} \|\nabla_{e_{a_1}}\vec{v} \wedge \nabla_{e_{a_2}}\vec{v}\|^2 + \cdots + \sum_{a_1<\cdots<a_{n-1}} \|\nabla_{e_{a_1}}\vec{v} \wedge \cdots \wedge \nabla_{e_{a_{n-1}}}\vec{v}\|^2\Big)^{\frac{1}{2}} \nu.$$

Note that $\operatorname{vol}(\vec{v}) \geq \operatorname{vol}(M)$ and also that only parallel fields attain the trivial minimum.

For odd-dimensional spheres, vector fields homologous to the Hopf fibration $\vec{v}_H$ have been studied, see [10], [3], [9] and [2]. In [5], a non-trivial lower bound of the volume of unit vector fields on spaces of constant curvature was obtained. In $\mathbb{S}^{2k+1}$, only the vector field $\vec{n}$ tangent to the geodesics from a fixed point (with two singularities) attains the volume of that bound. We call this field $\vec{n}$ north-south or radial vector field. We notice that unit vector fields with singularities show up in a natural way, see also [12].

For manifolds of dimension 5, a theorem showing how the topology of a vector field influences its volume appears in [4]. More precisely, the result in [4] is an inequality relating the volume of $\vec{v}$ and the Euler form of the orthogonal distribution to $\vec{v}$.

The purpose of this paper is to establish a relationship between the volume of unit vector fields and the indices of those fields around isolated singularities.

We consider these notes to be a preliminary effort to understand this phenomenon. For this reason, we have chosen a simple model where such a relationship is found. We hope this could serve as inspiration for more complex situations to be treated in a near future.

Precisely, we prove here:

THEOREM 1.1. — *Let $W = \mathbb{S}^n \setminus \{N, S\}$, $n = 2$ or 3, be the standard Euclidean sphere where two antipodal points N and S are removed. Let $\vec{v}$ be a unit smooth vector field defined on W. Then,*

$$\textit{for } n = 2, \quad \operatorname{vol}(\vec{v}) \geq \tfrac{1}{2}\big(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2\big)\operatorname{vol}(\mathbb{S}^2);$$

$$\textit{for } n = 3, \quad \operatorname{vol}(\vec{v}) \geq \big(|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|\big)\operatorname{vol}(\mathbb{S}^3),$$

where $I_{\vec{v}}(P)$ stands the Poincaré index of $\vec{v}$ around P.

It is easy to verify that the north-south field $\vec{n}$ achieves the equalities in the theorem. In fact, the volume of $\vec{n}$ in $\mathbb{S}^2$ is equal to $\frac{1}{2}\pi\mathrm{vol}(\mathbb{S}^2)$, and in $\mathbb{S}^3$ is $2\mathrm{vol}(\mathbb{S}^3)$. We have to point out that $\mathrm{vol}(\vec{n}) = \mathrm{vol}(\vec{v}_H)$ in $\mathbb{S}^3$.

The lower bound in $\mathbb{S}^3$ when the singularities are trivial (i.e. $I_{\vec{v}}(N) = I_{\vec{v}}(S) = 0$) has no special meaning.

We will comment briefly some possible extensions for this result in Section 3 of this paper.

2. Proof of the theorem

A key ingredient in the proof of the theorem is the application of the following result of Chern [7]. The second part of this statement is a special case of the result of Section 3 of that article.

Proposition 2.1 (see Chern [7]). — *Let M^n be an orientable Riemannian manifold of dimension n, with Riemannian connection 1-form ω and curvature form Ω. Then, there is an $(n-1)$-form Π on the unit tangent bundle T^1M with $\pi : T^1M \to M$ the bundle projection, so that:*

$$\mathrm{d}\Pi = \begin{cases} e(\Omega) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In addition, $\int_{\pi^{-1}(x)} \Pi = 1$ for any $x \in M$, that is, $\Pi_{|\pi^{-1}(x)}$ is the induced volume form of the fiber $\pi^{-1}(x)$, normalized to have volume 1.

The form Π as described by Chern is somewhat complicated. First, define forms ϕ_k for $k \in \{0, \dots, [\frac{1}{2}n] - 1\}$, by choosing a frame $\{e_1, \dots, e_n\}$ of TM, so that $\{e_1, \dots, e_{n-1}\}$ frame $\pi^{-1}(x)$ at $e_n \in \pi^{-1}(x)$. Then, at $e_n \in T^1M$,

$$\phi_k = \sum_{1 \le \alpha_1, \dots, \alpha_{n-1} \le n-1} \epsilon_{\alpha_1 \dots \alpha_{n-1}} \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}\alpha_{2k}} \wedge \omega_{\alpha_{2k+1} n} \wedge \cdots \wedge \omega_{\alpha_{n-1} n},$$

where $\epsilon_{\alpha_1 \dots \alpha_{n-1}}$ is the sign of the permutation, and from this

$$\Pi = \begin{cases} \dfrac{1}{\pi^{\frac{1}{2}n}} \displaystyle\sum_{k=0}^{\frac{1}{2}n-1} \frac{(-1)^k}{1 \cdot 3 \cdots (n-2k-1) \cdot 2^{k+\frac{1}{2}n} k!} \phi_k & \text{if } n \text{ is even,} \\ \dfrac{1}{2^n \pi^{\frac{1}{2}(n-1)} (\frac{1}{2}(n-1))!} \displaystyle\sum_{k=0}^{\frac{1}{2}(n-1)} (-1)^k \binom{\frac{1}{2}(n-1)}{k} \phi_k & \text{if } n \text{ is odd.} \end{cases}$$

Subsequent treatments of this general theory [8], [11] use more elegant formulations of forms similar to this, but usually only for the bundle of frames, and avoid the case where M is odd-dimensional.

The cases relevant to this research are for $n = 2$ and $n = 3$, where these formulas simplify to

$$\Pi = \begin{cases} \frac{1}{2\pi}\omega_{12} & \text{if } n = 2, \\ \frac{1}{4\pi}\left(\omega_{13} \wedge \omega_{23} - \Omega_{12}\right) & \text{if } n = 3. \end{cases}$$

Even though there is a common line of reasoning in the proof of both parts of the theorem, each dimension has its special features. For that reason, we provide separate proofs for dimensions 2 and 3.

2.1. Case $n = 2$. — Denote by g the usual metric on $\mathbb{S}^2$ induced from $\mathbb{R}^3$. Without loss of generality we take $N = (0, 0, 1)$ and $S = (0, 0, -1)$. On W we consider an oriented orthonormal local frame $\{e_1, e_2 = \vec{v}\}$. Its dual basis is denoted by $\{\theta_1, \theta_2\}$ and the connection 1-forms of ∇ are $\omega_{ij}(X) = g(\nabla_X e_j, e_i)$ for $i, j = 1, 2$ where X is a vector in the corresponding tangent space. In dimension 2, the volume (1) reduces to:

$$\operatorname{vol}(\vec{v}) = \int_{\mathbb{S}^2} \sqrt{1 + k^2 + \tau^2}\,\nu,$$

where $k = g(\nabla_{\vec{v}}\,\vec{v}, e_1)$ is the geodesic curvature of the integral curves of $\vec{v}$ and $\tau = g(\nabla_{e_1}\vec{v}, e_1)$ is the geodesic curvature of the curves orthogonal to $\vec{v}$. Also,

$$\omega_{12} = \tau\theta_1 + k\theta_2.$$

The first goal is to relate the integrand of the volume with the connection form ω_{12}. If S^1_φ is the parallel of $\mathbb{S}^2$ at latitude $\varphi \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ consider the unit field $\vec{u}$ on S^1_φ such that $\{\vec{u}, \vec{n}\}$ is positively oriented where $\vec{n}$ is the field pointing toward N. Let $\alpha \in [0, 2\pi]$ be the oriented angle from $\vec{u}$ to $\vec{v}$. Then $\vec{u} = \sin\alpha\, e_1 + \cos\alpha\,\vec{v}$. If $i : S^1_\varphi \to \mathbb{S}^2$ is the inclusion map, we have

$$i^*\omega_{12}(\vec{u}) = \tau\theta_1(\vec{u}) + k\theta_2(\vec{u}) = \tau\sin\alpha + k\cos\alpha. \tag{2}$$

We split the domain of the integral in northern and southern hemisphere, H^+ and H^- respectively. First we consider the northern hemisphere H^+. From the general inequality $\sqrt{a^2 + b^2} \geq |a\cos\beta + b\sin\beta| \geq a\cos\beta + b\sin\beta$, for any $a, b, \beta \in \mathbb{R}$, we have:

$$\begin{aligned} \sqrt{1 + k^2 + \tau^2} &\geq \cos\varphi + \sqrt{k^2 + \tau^2}\sin\varphi \\ &\geq \cos\varphi + \bigl|k\cos\alpha + \tau\sin\alpha\bigr|\sin\varphi = \cos\varphi + \bigl|i^*\omega_{12}(\vec{u})\bigr|\sin\varphi. \end{aligned} \tag{3}$$