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THE WKB METHOD AND GEOMETRIC INSTABILITY FOR NONLINEAR SCHRÖDINGER EQUATIONS ON SURFACES

BY LAURENT THOMANN

ABSTRACT. — In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation on a Riemannian surface which has a stable geodesic. These approximate solutions will lead to some instability properties of the equation.

RÉSUMÉ (*Méthode WKB et instabilité géométrique pour les équations de Schrödinger non linéaires sur des surfaces*)

À l'aide de la méthode WKB nous construisons des solutions approchées à l'équation de Schrödinger cubique sur une variété qui possède une géodésique stable. Cette construction permet d'obtenir des résultats d'instabilités dans des espaces de Sobolev.

1. Introduction

Let (M,g) be a Riemannian surface (i.e., a Riemannian manifold of dimension 2), orientable or not. We assume that M is either compact or a compact perturbation of the euclidian space, so that the Sobolev embeddings are true. Consider $\Delta = \Delta_g$ the Laplace-Beltrami operator. In this paper we are interested in constructing WKB approximations for the nonlinear cubic Schrödinger equation

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$$
(1) \qquad \begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = \varepsilon |u|^2 u(t,x), \quad \varepsilon = \pm 1, \\ u(0,x) = u_0(x) \in H^\sigma(M), \end{cases}
$$

that is, given a small parameter $0 < h < 1$ and an integer N, functions $u_N(h)$ satisfying

$$
(2) \qquad i\partial_t u_N(h) + \Delta u_N(h) = \varepsilon |u_N(h)|^2 u_N(h) + R_N(h),
$$

with $\|u_N(h)\|_{H^\sigma} \sim 1$ and $\|R_N(h)\|_{H^\sigma} \leq C_N h^N$.
Here h is introduced so that $u_N(h)$ oscillates with frequency $\sim \frac{1}{h}$.
These approximate solutions to (1) will lead to some instability properties in the following sense (where h^{-1} will play the role of n):

Definition 1.1. — We say that the Cauchy problem (1) is unstable near 0 in $H^\sigma(M)$, if for all $C > 0$ there exist times $t_n \longrightarrow 0$ and $u_{1,n}, u_{2,n} \in H^\sigma(M)$ solutions of (1) so that

$$
\begin{aligned}
\|u_{1,n}(0)\|_{H^\sigma(M)}, \ \|u_{2,n}(0)\|_{H^\sigma(M)} &\leq C, \\
\|u_{1,n}(0) - u_{2,n}(0)\|_{H^\sigma(M)} &\longrightarrow 0, \\
\limsup \|u_{1,n}(t_n) - u_{2,n}(t_n)\|_{H^\sigma(M)} &\geq \frac{1}{2}C,
\end{aligned}
$$

when $n \longrightarrow +\infty$.

This means that the problem is not uniformly well-posed, if we refer to the following definition:

Definition 1.2. — Let $\sigma \in \mathbb{R}$. Denote by $B_{R,\sigma}$ the ball of radius R in H^σ. We say that the Cauchy problem (1) is uniformly well-posed in H^σ if the flow map

$$
u_0 \in B_{R,\sigma} \cap H^1(M) \longmapsto \Phi_t(u_0) \in H^\sigma(M),
$$

is uniformly continuous for any t.

We now state our instability result:

Proposition 1.3. — *Let $0 < \sigma < \frac{1}{4}$, and assume that M has a stable and non degenerated periodic geodesic (see Assumptions 1 and 2), then the Cauchy problem* (1) *is not uniformly well-posed.*

This problem is motivated by the following results: Let (M,g) be a riemannian compact surface, then in [5], N. Burq, P. Gérard and N. Tzvetkov prove that (1) is uniformly well-posed in $H^\sigma(M)$ for $\sigma > \frac{1}{2}$. Whereas, in [4], they show that (1) is unstable on the sphere $\mathbb{S}^2$ for $0 < \sigma < \frac{1}{4}$. In fact they construct solutions of (1) of the form

$$
(3) \qquad u_n^\kappa(t,x) = \kappa e^{i\lambda_n^\kappa t}(n^{\frac{1}{4}-\sigma}\psi_n(x) + r_n(t,x)),
$$

where $0 < \kappa < 1$, $\psi_n = (x_1 + ix_2)^n$ is a spherical harmonic which concentrates on the equator of the sphere when $n \longrightarrow +\infty$ and where r_n is an error term which is small. To obtain instability, they consider $\kappa_n \longrightarrow \kappa$, then

$$\|u_n^\kappa(0) - u_n^{\kappa_n}(0)\|_{H^\sigma(\mathbb{S}^2)} \lesssim |\kappa - \kappa_n| \longrightarrow 0,$$

but

$$\|u_n^\kappa(t_n) - u_n^{\kappa_n}(t_n)\|_{H^\sigma(\mathbb{S}^2)} \gtrsim \kappa|\mathrm{e}^{i\lambda_n^\kappa t_n} - \mathrm{e}^{i\lambda_n^{\kappa_n} t_n}| \longrightarrow 2\kappa,$$

with a suitable choice of $t_n \longrightarrow 0$.
We follow this strategy but as the surface is not rotation invariant, the ansatz will be more complicated than (3).
This result is sharp, because in [6] they show that (1) is uniformly well-posed on $\mathbb{S}^2$ when $\sigma > \frac{1}{4}$.
On the other hand, in [3] J. Bourgain shows that (1) is uniformly well-posed on the rational torus $\mathbb{T}^2$ when $\sigma > 0$.
These results show how the geometry of M can lead to instability for the equation (1). Therefore it seems reasonable to obtain a result like Proposition 1.3 with purely geometric assumptions.

We first make the following assumption on M:

Assumption 1. — *The manifold M has a periodic geodesic.*

Denote by γ such a geodesic, then there exists a system of coordinates (s, r) near γ, say for $(s, r) \in \mathbb{S}^1 \times]-r_0, r_0[$, called Fermi coordinates such that (see [13], p. 80)

1. The curve $r = 0$ is the geodesic γ parametrized by arclength and
2. The curves $s =$ constant are geodesics parametrized by arclength. The curves $r =$ constant meet these curves perpendicularly.
3. In this system the metric writes

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a^2(s, r) \end{pmatrix}.$$

We set the length of γ equal to 2π. Denote by $R(s, r)$ the Gauss curvature at (s, r), then a is the unique solution of

$$\begin{cases} \dfrac{\partial^2 a}{\partial r^2} + R(s, r)a = 0, \\ a(s, 0) = 1, \ \dfrac{\partial a}{\partial r}(s, 0) = 0. \end{cases} \tag{4}$$

The initial conditions traduce the fact that the curve $r = 0$ is a unit-speed geodesic. In these coordinates the Laplace-Beltrami operator is

$$\Delta := \frac{1}{\sqrt{\det g}}\mathrm{div}(\sqrt{\det g}\, g^{-1}\nabla) = \frac{1}{a}\partial_s(\frac{1}{a}\partial_s) + \frac{1}{a}\partial_r(a\partial_r).$$

A function on M, defined locally near γ, can be identified with a function of $[0,2\pi]\times]-r_0,r_0[$ such that

$$\forall (s,r) \in [0,2\pi]\times]-r_0,r_0[\quad f(s+2\pi,r) = f(s,\omega r)$$

where $\omega = 1$ if M is orientable and $\omega = -1$ if M is not. Define

$$\omega_1 = \frac{1}{2}(\omega - 1) \in \{-1, 0\}. \tag{5}$$

From (4) we deduce that a admits the Taylor expansion

$$a = 1 - \frac{1}{2}R(s)r^2 + R_3(s)r^3 + \cdots + R_p(s)r^p + o(r^p), \tag{6}$$

with $R(s) = R(s,0)$ and

$$R_k(s) = \frac{1}{k!}\frac{\partial^k a}{\partial r^k}(s,0), \tag{7}$$

for $k \geq 3$.
As $a(s+2\pi,r) = a(s,\omega r)$, we deduce $R(s+2\pi) = R(s)$ and for all $j \geq 3$, $R_j(s+2\pi) = \omega^j R_j(s)$.
Let $p_2 = \frac{1}{a^2}\sigma^2 + \rho^2$ be the principal symbol of Δ, and

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}s(t) = \dfrac{\partial p_2}{\partial \sigma} = \dfrac{2\sigma}{a^2}, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = -\dfrac{\partial p_2}{\partial s} = -\partial_s(\dfrac{1}{a^2})\sigma^2, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}r(t) = \dfrac{\partial p_2}{\partial \rho} = 2\rho, \ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\dfrac{\partial p_2}{\partial r} = -\partial_r(\dfrac{1}{a^2})\sigma^2, \\ s(0) = s_0, \ \sigma(0) = \sigma_0, r(0) = r_0, \ \rho(0) = \rho_0, \end{cases} \tag{8}$$

its associated hamiltonian system, where $p_2 = p_2(s(t), r(t), \sigma(t), \rho(t))$. The system (8) admits a unique solution and defines the hamiltonian flow

$$\Phi_t : (s_0, \sigma_0, r_0, \rho_0) \longmapsto (s(t), \sigma(t), r(t), \rho(t)).$$

The curve $\Gamma = \{(s(t) = t, \sigma(t) = 1/2, r(t) = 0, \rho(t) = 0), t \in [0,2\pi]\}$ is solution of (8) and its projection in the (s,r) space is the curve γ. Now denote by ϕ the Poincaré map associated to the trajectory Γ and to the hyperplane $\Sigma = \{s = 0\}$. There exists a neighborhood $\mathcal{N}$ of $(\sigma = 1/2, r = 0, \rho = 0)$ such that the following makes sense: solve the system (8) with the initial conditions $(0, \sigma_0, r_0, \rho_0) \in \{0\} \times \mathcal{N}$ and let T be such that $s(T) = 2\pi$, then ϕ is the application

$$\phi : (r_0, \rho_0) \longmapsto (r(T), \rho(T)).$$