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Matei Toma

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# ON THE KÄHLER RANK OF COMPACT COMPLEX SURFACES

BY MATEI TOMA

ABSTRACT. — Harvey and Lawson introduced the Kähler rank and computed it in connection to the cone of positive exact currents of bidimension $(1,1)$ for many classes of compact complex surfaces. In this paper we extend these computations to the only further known class of surfaces not considered by them, that of Kato surfaces. Our main tool is the reduction to the dynamics of associated holomorphic contractions $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$.

RÉSUMÉ (*Sur le rang de Kähler des surfaces complexes compactes*)

Harvey et Lawson ont introduit et calculé le rang de Kähler en relation avec le cône des courants positifs fermés de bidimension $(1,1)$ pour beaucoup de classes de surfaces complexes compactes. Dans ce travail nous étendons ces calculs à la seule classe de surfaces connues et qui n'avait pas été considérée par eux, celle des surfaces de Kato. Notre outil principal est la réduction à la dynamique des contractions holomorphes $(\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$ associées.

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MATEI TOMA, Institut de Mathématiques Élie Cartan, Nancy-Université, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France and Institute of Mathematics of the Romanian Academy • *E-mail :* `toma@iecn.u-nancy.fr` • *Url :* `http://www.iecn.u-nancy.fr/~toma/`

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## 1. Introduction

In [8] Harvey and Lawson give a characterisation of Kählerianity for compact complex surfaces in terms of existence (or rather non-existence) of closed positive currents which are $(1,1)$-components of a boundary. The authors also investigate and describe the cones formed by such currents for many types of non-Kähler surfaces: elliptic, Hopf, Inoue. Later Lamari proved that every non-Kähler surface admits non-trivial positive $d$-exact currents of bidimension $(1,1)$; cf. [10]. In order to estimate the degree of non-Kählerianity of a compact complex surface smooth positive $d$-exact currents are considered in [8] and the Kähler rank is defined as follows: the Kähler rank is *two* if the surface admits some Kähler metric, *one* if it admits some positive $d$-exact $(1,1)$-form with some supplementary property and *zero* in the remaining case; see the more precise Definition 2. Unfortunately it is not clear whether the Kähler rank is a bimeromorphic invariant.

We consider in this paper instead a bimeromorphic invariant which we call the *modified Kähler rank* and which we define to be *two* if the surface is Kähler, *one* if the cone of positive $d$-exact currents of bidimension $(1,1)$ is larger than a half-line and *zero* if this cone is a half-line. The two notions agree in the cases considered in [8]. Our main result is the computation of the cones of positive exact $(1,1)$-currents for Kato surfaces. These are surfaces whose minimal models have positive second Betti number and admit a global spherical shell (see Definition 9) and are the only "known" compact complex surfaces not considered in [8]. (We refer the reader to [1] for the general theory of compact complex surfaces.) It turns out that the modified Kähler rank does not coincide with the Kähler rank in general. In order to perform our computations we reduce ourselves to the investigation of plurisubharmonic functions with a certain invariance property with respect to polynomial automorphisms of $\mathbb{C}^2$ associated to Kato surfaces. As a corollary we obtain

THEOREM 1. — *(a) Every positive d-exact* $(1,1)$*-current on a Kato surface, and more generally on any "known" non-Kählerian compact complex surface, is a foliated current for some holomorphic foliation of the surface.*

*(b) All positive d-closed* $(1,1)$*-currents on the "known" non-Kählerian compact complex surfaces excepting on parabolic Inoue surfaces are foliated currents for some holomorphic foliations.*

See Section 3 for definitions.

## 2. The Kähler rank

We let $X$ be a compact complex surface and denote by $P_{bdy} = P_{bdy}(X)$, $P^{\infty}_{bdy} = P^{\infty}_{bdy}(X)$ the cones of positive currents of bidimension $(1,1)$ which are boundaries (i.e. are $d$-exact), respectively smooth such currents. These objects have been first studied in [8]. In [10] it was shown that $X$ is Kähler if and only if $P_{bdy}(X)$ is trivial. It is easy to see that on a non-Kähler surface every positive $d$-closed differential form of type $(1,1)$ is $d$-exact, cf. [10]. The following definition of Harvey and Lawson gives a measure of non-Kählerianity by looking at the positive closed differential$(1,1)$-forms on $X$. It is roughly speaking the largest generic rank of a positive closed $(1,1)$-form on $X$.

Definition 2. — *Let $B(X) = \{x \in X \mid \exists \phi \in P^{\infty}_{bdy}(X)\ \phi_x \neq 0\}$. The* Kähler rank *of $X$ is defined to be* two *if $X$ admits a Kähler metric. If $X$ is non-Kähler the Kähler rank is set to be* one *when $B(X)$ contains a non-trivial Zariski open subset of $X$ and* zero *otherwise.*

It is not known whether the Kähler rank is a bimeromorphic invariant.

We propose also the following:

Definition 3. — *The* modified Kähler rank *of $X$ is defined to be* two *when $X$ admits no non-trivial positive exact current of bidimension $(1,1)$,* zero *when it admits exactly one such current up to a multiplicative constant and* one *otherwise.*

One can show easily that the modified Kähler rank is a bimeromorphic invariant by taking push-down and pull-back of currents through blowing up maps. See the proof of Proposition 7 for a more precise description.

For elliptic surfaces, primary Hopf surfaces and Inoue surfaces one sees that the Kähler rank and the modified Kähler rank coincide using the precise description of $P_{bdy}$ given in [8] in these cases.

Proposition 4. — *Let $T$ be a positive exact current of bidimension $(1,1)$ on the compact complex surface $X$. Then there is a representation $\rho : \pi_1(X) \to (\mathbb{R},+)$ and a plurisubharmonic function $u$ on the universal cover $\tilde{X}$ of $X$ such that $T = dd^c u$ and $u \circ g = u + \rho(g)$ for all $g \in \pi_1(X)$.*

*The function $u$ can be chosen to be smooth if $T$ is smooth.*

*Proof.* — One has $b_1 = \dim_{\mathbb{C}} H^1(X,\mathbb{C}) = \dim_{\mathbb{C}} H^1(X,\mathcal{O}_X) + \dim_{\mathbb{C}} H^0(X,\Omega^1_X) = h^{0,1} + h^{1,0}$, cf [1]. Denoting the sheaf of closed differential $(1,0)$-forms by $d\mathcal{O}_X$ and looking at the long exact cohomology sequence of

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to d\mathcal{O}_X \to 0$$

one gets an exact sequence

$$0 \to H^0(d\mathcal{O}_X) \to H^1(\mathbb{C}_X) \to H^1(\mathcal{O}_X).$$

By the above equality follows now the surjectivity of the natural map $H^1(\mathbb{C}_X) \to H^1(\mathcal{O}_X)$. This is given by mapping a de Rham cohomology class $[\beta]$ of a differential form $\beta = \beta^{1,0} + \beta^{0,1}$ onto the Dolbeault cohomology class of $\beta^{0,1}$.

Let now $T$ be a positive exact current of bidimension $(1,1)$ on $X$. Then $T = dS$ with $S = S^{1,0} + S^{0,1}$, $S^{1,0}$, $S^{0,1}$ currents of order zero and bidegree $(1,0)$ and $(0,1)$ respectively, and $S^{1,0} = \bar{S}^{0,1}$. Since $\bar\partial S^{0,1} = 0$, $S^{0,1}$ represents a cohomology class in $H^1(\mathcal{O}_X)$ and let $\beta = \beta^{1,0} + \beta^{0,1}$ be a closed differential form with $[\beta^{0,1}] = [S^{0,1}]$ in $H^1(\mathcal{O}_X)$ and $U$ a current of degree 0 on $X$ with $S^{0,1} = \beta^{0,1} + \bar\partial U$. The lift $\tilde\beta$ of $\beta$ to the universal cover $\tilde X$ is $d$-exact and let $f$ be a smooth function on $\tilde X$ with $df = \beta$. In particular $\bar\partial f = \beta^{0,1}$. This implies $S^{0,1} = \bar\partial(f + U)$ and $T = dS = d(\bar\partial(f+U) + \partial(\bar f + \bar U)) = i\partial\bar\partial(2\mathfrak{Im}\,(f+U))$.

Moreover for $g \in \pi_1(X)$ we have $d(f \circ g - f) = 0$ hence $f \circ g - f$ must be constant. Set $\rho(g) = 2\mathfrak{Im}\,(f\circ g - f)$. The current $2\mathfrak{Im}\,(f+U)$ is associated to a plurisubharmonic function $u$ on $\tilde X$. Since $u - 2\mathfrak{Im}\, f = 2\mathfrak{Im}\, U$ comes from $X$ we see that $u$ has the desired automorphy behaviour with respect to the action of $\pi_1(X)$.

It is clear that $u$ can be chosen to be smooth when $T$ is smooth. □

Definition 5. — *We say that an effective reduced divisor $C = C_1 + \cdots + C_n$ on $X$ is a* cycle of rational curves *if $n \geq 1$, $C_1, \ldots, C_n$ are rational curves and either $n = 1$ and $C_1$ has a node or $n > 1$, all components $C_1, \ldots, C_n$ are smooth and the dual graph of $C$ is cyclic.*

Corollary 6. — *For a compact complex surface $X$ with a cycle of rational curves $C$ and $b_1(X) = 1$ the Kähler rank is zero.*

*Proof.* — Under the above hypotheses the natural map $\mathbb{Z} \cong \pi_1(C) \to \pi_1(X)$ is an isomorphism by a Theorem of Nakamura, [13]. Let $g$ be a generator of $\pi_1(X)$. If the Kähler rank of $X$ were one, we would get a smooth non-constant plurisubharmonic function $u$ on the universal cover $\tilde X$ satisfying

$$u \circ g = u + c$$

for some constant $c \in \mathbb{R}$ by Proposition 4. The inverse image $\tilde C$ of the cycle of rational curves $C$ of $X$ is an infinite chain of rational curves on $\tilde X$. Since $u$ is smooth and constant on each link of this chain we would get $c = 0$ hence $u$ would descend to $X$. But here $u$ must be constant contradicting our assumptions. □

Proposition 7. — *The modified Kähler rank is a bimeromorphic invariant.*